MABUCHI AND AUBIN-YAU FUNCTIONALS OVER COMPLEX MANIFOLDS

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ABSTRACT. In the previous papers [2, 3] the author constructed Mabuchi and Aubin-Yau functionals over any complex surfaces and three-folds, respectively. Using the method in [3], we construct those functionals over any complex manifolds of the complex dimension bigger than or equal to 2.

CONTENTS

1. INTRODUCTION

Mabuchi and Aubin-Yau functionals play a crucial role in studying Kähler-Einstein metrics and constant scalar curvatures(see [5]). How to generalize these functionals from Kähler geometry to complex geometry is an interesting problem:

Question 1.1. *Can we define Mabuchi and Aubin-Yau functionals over compact complex manifolds so that these functionals coincide with the original definitions and satisfy the same basic properties?*

In [2, 3], the author answered this question in the complex dimension two and three, respectively, and proved similar results in the Kähler setting. By carefully checking and using a similar method in [3], we can construct those functionals in higher dimension cases. So, now, we give an affirmative answer to Question 1.1.

1.1. **Mabuchi and Aubin-Yau functionals on Kähler manifolds.** In this subsection we review Mabuchi and Aubin-Yau functionals on Kähler manifolds, and describe some basic properties of these functionals which also hold in any complex manifolds. Let (X,ω) be a compact Kähler manifold of the complex dimension n. Then the volume

$$V_\omega := \int_X \omega^n \tag{1.1}$$

depends only on the Kähler class of ω. Let $\mathcal{P}^{\rm Kahler}_\omega$ denote the space of Kähler potentials and define the Mabuchi functional, for any smooth functions $\varphi',\varphi''\in\mathcal{P}^{\rm Kahler}_\omega$, by

$$\mathcal{L}^{\rm M,Kahler}_\omega(\varphi',\varphi'') := \frac{1}{V_\omega}\int_0^1\int_X \dot{\varphi}_t\omega^n_{\varphi_t}dt \tag{1.2}$$

where φ_t is any smooth path in $\mathcal{P}^{\rm Kahler}_\omega$ from φ' to φ''. Mabuchi [4] showed that (1.2) is well-defined.

Using (1.2) we can define Aubin-Yau functionals, for any smooth function $\varphi\in\mathcal{P}^{\rm Kahler}_\omega$, as follows:

$$\mathcal{I}^{\rm AY,Kahler}_\omega(\varphi) = \frac{1}{V_\omega}\int_X\varphi(\omega^n-\omega^n_\varphi), \tag{1.3}$$

$$\mathcal{J}^{\rm AY,Kahler}_\omega(\varphi) = -\mathcal{L}^{\rm M,Kahler}_\omega(0,\varphi)+\frac{1}{V_\omega}\int_X\varphi\omega^n. \tag{1.4}$$

So Aubin-Yau functionals are also well-defined. The basic and often useful inequalities, for any smooth function $\varphi\in\mathcal{P}^{\rm Kahler}_\omega$, are

$$\frac{n}{n+1}\mathcal{I}^{\rm AY,Kahler}_\omega(\varphi)-\mathcal{J}^{\rm AY,Kahler}_\omega(\varphi) \geq 0, \tag{1.5}$$

$$(n+1)\mathcal{J}^{\rm AY,Kahler}_\omega(\varphi)-\mathcal{I}^{\rm AY,Kahler}_\omega(\varphi) \geq 0. \tag{1.6}$$

An important consequence is that we will use the inequalities (1.5) and (1.6) to determine the extra terms on the definitions of $\mathcal{I}^{\rm AY}_\omega(\varphi)$ and $\mathcal{J}^{\rm AY}_\omega(\varphi)$, which are Aubin-Yau functionals over complex manifolds.

However, if ω is not closed, then the above definitions (1.2), (1.3), and (1.4) do not make any sense. Hence we should add some extra terms on the definitions of those functionals; these extra terms should involve $\partial\omega$ and $\overline{\partial}\omega$, but, the essential question is to find the structure of the extra terms. Roughly speaking, $\mathcal{I}^{\rm AY}_\omega(\varphi)$ and $\mathcal{J}^{\rm AY}_\omega(\varphi)$ can be written as

$$\mathcal{I}^{\rm AY}_\omega(\varphi) = \mathcal{I}^{\rm AY,Kahler}_\omega(\varphi)+\text{terms involving } \partial\omega,\overline{\partial}\varphi+\text{terms invloving } \overline{\partial}\omega,\partial\varphi,$$

$$\mathcal{J}^{\rm AY}_\omega(\varphi) = \mathcal{J}^{\rm AY,Kahler}_\omega(\varphi)+\text{terms involving } \partial\omega,\overline{\partial}\varphi+\text{terms involving } \overline{\partial}\omega,\partial\varphi.$$

In the following sections, we will explicitly determine the extra terms.

Throughout the rest part of this paper, we denote by (X,g) a compact complex manifold of the complex dimension $n\geq 2$, and ω be the associated real $(1,1)$-form. Let

$$\mathcal{P}_\omega := \left\{\varphi\in C^\infty(X)_{\mathbb{R}}\middle|\omega_\varphi := \omega+\sqrt{-1}\partial\overline{\partial}\varphi>0\right\} \tag{1.7}$$

be the space of all real-valued smooth functions on X whose associated real $(1,1)$-forms are positive.

1.2. **Mabuchi and Aubin-Yau functionals on complex surfaces.** In this subsection we recall the main result in [2]. Let (X, g) be a compact complex manifold of the complex dimension 2 and ω be its associated real $(1,1)$-form. For any $\varphi', \varphi'' \in \mathcal{P}_\omega$, we define

$$\begin{aligned}\mathcal{L}^{\mathrm{M}}_\omega(\varphi', \varphi'') &:= \frac{1}{V_\omega}\int_0^1\int_X \dot{\varphi}_t \cdot \omega^2_{\varphi_t} dt \\ &- \frac{1}{V_\omega}\int_0^1\int_X \sqrt{-1}\partial\omega \wedge (\overline{\partial}\dot{\varphi}_t \cdot \varphi_t) dt \\ &+ \frac{1}{V_\omega}\int_0^1\int_X \sqrt{-1}\overline{\partial}\omega \wedge (\partial\dot{\varphi}_t \cdot \varphi_t) dt,\end{aligned} \tag{1.8}$$

where $\{\varphi_t\}_{0\le t\le 1}$ is any smooth path in $\mathcal{P}_\omega$ from φ' to φ''. Then in [2] we showed that the functional (1.8) is independent of the choice of the smooth path $\{\varphi_t\}_{0\le t\le 1}$. If we set

$$\mathcal{L}^{\mathrm{M}}_\omega(\varphi) := \mathcal{L}^{\mathrm{M}}_\omega(0, \varphi),$$

then we have an explicit formula [2] of $\mathcal{L}^{\mathrm{M}}_\omega(\varphi)$:

$$\begin{aligned}\mathcal{L}^{\mathrm{M}}_\omega(\varphi) &= \frac{1}{3V_\omega}\int_X \varphi(\omega^2 + \omega\wedge\omega_\varphi + \omega^2_\varphi) \\ &+ \frac{1}{2V_\omega}\int_X \varphi\left(-\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\right).\end{aligned} \tag{1.9}$$

Now Aubin-Yau functionals are defined by

$$\begin{aligned}\mathcal{I}^{\mathrm{AY}}_\omega(\varphi) &:= \frac{1}{V_\omega}\int_X \varphi(\omega^2 - \omega^2_\varphi) \\ &- \frac{1}{V_\omega}\int_X \varphi\cdot\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \frac{1}{V_\omega}\int_X \varphi\cdot\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi,\end{aligned} \tag{1.10}$$

$$\begin{aligned}\mathcal{J}^{\mathrm{AY}}_\omega(\varphi) &:= -\mathcal{L}^{\mathrm{M}}_\omega(\varphi) + \frac{1}{V_\omega}\int_X \varphi\omega^2 \\ &- \frac{1}{V_\omega}\int_X \varphi\cdot\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \frac{1}{V_\omega}\int_X \varphi\cdot\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi.\end{aligned} \tag{1.11}$$

Moreover they also satisfy the inequalities (1.5) and (1.6); that is

$$\begin{aligned}\frac{2}{3}\mathcal{I}^{\mathrm{AY}}_\omega(\varphi) - \mathcal{J}^{\mathrm{AY}}_\omega(\varphi) &\ge 0, \\ 3\mathcal{J}^{\mathrm{AY}}_\omega - \mathcal{I}^{\mathrm{AY}}_\omega(\varphi) &\ge 0.\end{aligned} \tag{1.12}$$

1.3. **Mabuchi and Aubin-Yau functionals on complex three-folds.** The functionals over complex three-folds are very different with these over complex surfaces. For any $\varphi', \varphi'' \in \mathcal{P}_\omega$, we define

$$
\begin{aligned}
\mathcal{L}^{\mathrm{M}}_\omega(\varphi', \varphi'') &:= \frac{1}{V_\omega}\int_0^1\int_X \dot{\varphi}_t \omega^3_{\varphi_t} dt \\
&- \frac{3}{V_\omega}\int_0^1\int_X \sqrt{-1}\partial\omega \wedge \omega_{\varphi_t} \wedge (\overline{\partial}\dot{\varphi}_t \cdot \varphi_t) dt \\
&+ \frac{3}{V_\omega}\int_0^1\int_X \sqrt{-1}\overline{\partial}\omega \wedge \omega_{\varphi_t} \wedge (\partial\dot{\varphi}_t \cdot \varphi_t) dt \\
&- \frac{1}{V_\omega}\int_0^1\int_X \partial\varphi_t \wedge \overline{\partial}\varphi_t \wedge \partial\omega \wedge \overline{\partial}\dot{\varphi}_t \\
&- \frac{1}{V_\omega}\int_0^1\int_X \overline{\partial}\varphi_t \wedge \partial\varphi_t \wedge \overline{\partial}\omega \wedge \partial\dot{\varphi}_t,
\end{aligned} \tag{1.13}
$$

where $\{\varphi_t\}_{0\leq t\leq 1}$ is any smooth path in $\mathcal{P}_\omega$ from φ' to φ''. In [3], we proved that (1.13) is well-defined.

For any $\varphi \in \mathcal{P}_\omega$ we also set $\mathcal{L}^{\mathrm{M}}_\omega(\varphi) := \mathcal{L}^{\mathrm{M}}_\omega(0, \varphi)$. If we chose $\varphi_t = t\cdot\varphi$, then we have an explicit formula [3] of $\mathcal{L}^{\mathrm{M}}_\omega(\varphi)$:

$$
\begin{aligned}
\mathcal{L}^{\mathrm{M}}_\omega(\varphi) &= \frac{1}{4V_\omega}\sum_{i=0}^{3}\int_X \varphi\omega^i_\varphi \wedge \omega^{3-i} \\
&- \sum_{i=0}^{1}\frac{i+1}{2V_\omega}\int_X \varphi\omega^i_\varphi \wedge \omega^{1-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&+ \sum_{i=0}^{1}\frac{i+1}{2V_\omega}\int_X \varphi\omega^i_\varphi \wedge \omega^{1-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi.
\end{aligned} \tag{1.14}
$$

Now we define Aubin-Yau functionals $\mathcal{I}^{\mathrm{AY}}_\omega, \mathcal{J}^{\mathrm{AY}}_\omega$ for any compact complex three-fold (X, ω):

$$
\begin{aligned}
\mathcal{I}^{\mathrm{AY}}_\omega(\varphi) &:= \frac{1}{V_\omega}\int_X \varphi(\omega^3 - \omega^3_\varphi) \\
&- \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi - \frac{3}{2V_\omega}\int_X \varphi\omega \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&+ \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi + \frac{3}{2V_\omega}\int_X \varphi\omega \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi,
\end{aligned} \tag{1.15}
$$

$$
\begin{aligned}
\mathcal{J}^{\mathrm{AY}}_\omega(\varphi) &:= -\mathcal{L}^{\mathrm{M}}_\omega(\varphi) + \frac{1}{V_\omega}\int_X \varphi\omega^3 \\
&- \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi - \frac{3}{2V_\omega}\int_X \varphi \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&+ \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi + \frac{3}{2V_\omega}\int_X \varphi\omega \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi.
\end{aligned} \tag{1.16}
$$

Similarly, we have

$$
\begin{aligned}
\frac{3}{4}\mathcal{I}^{\mathrm{AY}}_\omega(\varphi) - \mathcal{J}^{\mathrm{AY}}_\omega(\varphi) &\geq 0, \\
4\mathcal{J}^{\mathrm{AY}}_\omega(\varphi) - \mathcal{I}^{\mathrm{AY}}_\omega(\varphi) &\geq 0.
\end{aligned} \tag{1.17}
$$

1.4. **Mabuchi and Aubin-Yau functionals on complex manifolds.** In this subsection we assume that the complex dimension n of the compact complex manifold (X, ω) is bigger than or equal to 3. For $\varphi', \varphi'' \in \mathcal{P}_\omega$, define

$$
\begin{aligned}
(1.18)\mathcal{L}^{\mathrm{M}}_\omega(\varphi', \varphi'') \quad := \quad & \frac{1}{V_\omega}\int_0^1\int_X \dot{\varphi}_t \omega^n_{\varphi_t} dt \\
- \quad & \frac{1}{V_\omega}\int_0^1\int_X \frac{n(n-1)\sqrt{-1}}{2}\partial\omega \wedge \omega^{n-2}_{\varphi_t} \wedge (\overline{\partial}\dot{\varphi}_t \cdot \varphi_t) dt \\
+ \quad & \frac{1}{V_\omega}\int_0^1\int_X \frac{n(n-1)\sqrt{-1}}{2}\overline{\partial}\omega \wedge \omega^{n-2}_{\varphi_t} \wedge (\partial\dot{\varphi}_t \cdot \varphi_t) dt \\
+ \quad & \sum_{i=1}^{n-2}\frac{1}{V_\omega}\int_0^1\int_X (-1)^i\binom{n}{i+2}\partial\varphi_t \wedge \partial\omega \wedge \overline{\partial}\dot{\varphi}_t \wedge \overline{\partial}\varphi_t \wedge \omega^{n-i-2}_{\varphi_t} \wedge (\sqrt{-1}\partial\overline{\partial}\varphi_t)^{i-1} \\
+ \quad & \sum_{i=1}^{n-2}\frac{1}{V_\omega}\int_0^1\int_X (-1)^i\binom{n}{i+2}\overline{\partial}\varphi_t \wedge \overline{\partial}\omega \wedge \partial\dot{\varphi}_t \wedge \partial\varphi_t \wedge \omega^{n-i-2}_{\varphi_t} \wedge (\sqrt{-1}\partial\overline{\partial}\varphi_t)^{i-1},
\end{aligned}
$$

where $\{\varphi_t\}_{0\leq t\leq 1}$ is any smooth path in $\mathcal{P}_\omega$ from φ' to φ''. In Section 2, we prove

Theorem 1.2. *For any $n \geq 3$, the functional (1.18) is independent of the choice of the smooth path $\{\varphi_t\}_{0\leq t\leq 1}$ in $\mathcal{P}_\omega$ from φ' to φ''.*

As a consequence of Theorem 1.2, by taking $\varphi_t = t\cdot\varphi$, we have, for any $\varphi \in \mathcal{P}_\omega$,

$$
\begin{aligned}
\mathcal{L}^{\mathrm{M}}_\omega(\varphi) \quad := \quad \mathcal{L}^{\mathrm{M}}_\omega(0,\varphi) \quad = \quad & \frac{1}{V_\omega}\sum_{i=0}^{n}\int_X \frac{1}{n+1}\varphi\omega^i_\varphi \wedge \omega^{n-i} \\
- \quad & \sum_{i=0}^{n-2}\frac{i+1}{2V_\omega}\int_X \varphi\omega^i_\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
+ \quad & \sum_{i=0}^{n-2}\frac{i+1}{2V_\omega}\int_X \varphi\omega^i_\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi.
\end{aligned}
\tag{1.19}
$$

As before, we define Aubin-Yau functionals for any complex manifolds by

$$
\begin{aligned}
\mathcal{I}^{\mathrm{AY}}_\omega(\varphi) \quad := \quad & \frac{1}{V_\omega}\int_X \varphi(\omega^n - \omega^n_\varphi) - \frac{n}{2V_\omega}\sum_{i=0}^{n-2}\int_X \varphi\omega^i_\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
+ \quad & \frac{n}{2V_\omega}\sum_{i=0}^{n-2}\int_X \varphi\omega^i_\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
= \quad & \frac{1}{V_\omega}\sum_{i=0}^{n-1}\int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^i_\varphi \wedge \omega^{n-1-i},
\end{aligned}
\tag{1.20}
$$

$$
\begin{aligned}
\mathcal{J}^{\mathrm{AY}}_\omega(\varphi) \quad := \quad & -\mathcal{L}^{\mathrm{M}}_\omega(\varphi) + \frac{1}{V_\omega}\int_X \varphi\omega^n - \frac{n}{2V_\omega}\sum_{i=0}^{n-2}\int_X \varphi\omega^i_\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
+ \quad & \frac{n}{2V_\omega}\sum_{i=0}^{n-2}\int_X \varphi\omega^i_\varphi \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
= \quad & \frac{1}{V_\omega}\sum_{i=0}^{n-1}\int_X \frac{n-i}{n+1}\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^i_\varphi \wedge \omega^{n-1-i}.
\end{aligned}
\tag{1.21}
$$

In Section 3, we shall show the following

Theorem 1.3. *For any $\varphi \in \mathcal{P}_\omega$, one has*

$$\frac{n}{n+1}\mathcal{I}^{\rm AY}_\omega(\varphi) - \mathcal{J}^{\rm AY}_\omega(\varphi) \geq 0, \tag{1.22}$$

$$(n+1)\mathcal{J}^{\rm AY}_\omega(\varphi) - \mathcal{I}^{\rm AY}_\omega(\varphi) \geq 0. \tag{1.23}$$

In particular, $\mathcal{I}^{\rm AY}_\omega(\varphi), \mathcal{J}^{\rm AY}_\omega(\varphi)$ are nonnegative, and

$$\frac{1}{n+1}\mathcal{I}^{\rm AY}_\omega(\varphi) \leq \mathcal{J}^{\rm AY}_\omega(\varphi) \leq \frac{n}{n+1}\mathcal{I}^{\rm AY}_\omega(\varphi), \tag{1.24}$$

$$\frac{n+1}{n}\mathcal{J}^{\rm AY}_\omega(\varphi) \leq \mathcal{I}^{\rm AY}_\omega(\varphi) \leq (n+1)\mathcal{J}^{\rm AY}_\omega(\varphi), \tag{1.25}$$

$$\begin{aligned}\frac{1}{n}\mathcal{J}^{\rm AY}_\omega(\varphi) \leq \frac{1}{n+1}\mathcal{J}^{\rm AY}_\omega(\varphi) &\leq \mathcal{I}^{\rm AY}_\omega(\varphi) - \mathcal{J}^{\rm AY}_\omega(\varphi) \\ &\leq \frac{n}{n+1}\mathcal{I}^{\rm AY}_\omega(\varphi) \leq n\mathcal{J}^{\rm AY}_\omega(\varphi).\end{aligned} \tag{1.26}$$

1.5. **Further questions.** Up to now we consider functionals over compact complex manifolds without boundary, and we hope that the similar constructions can be achieved for compact complex manifolds with boundary.

Question 1.4. *Can we define Mabuchi and Aubin-Yau functionals over compact complex manifolds with boundary so that these functionals coincide with the original definitions and satisfy the same basic properties?*

There are other functionals, for example, Mabuchi $\mathcal{K}^{\rm M}_\omega$ [4] functional, Chen-Tian functionals [1], etc. We can ask the following

Question 1.5. *Can we define the analogy Mabuchi $\mathcal{K}^{\rm M}_\omega$ and Chen-Tian functionals over complex manifolds with(out) boundary?*

In the future, we will study those two questions.

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2. Mabuchi $\mathcal{L}^{\rm M}_\omega$ functional on complex manifolds

In this section we assume that (X, ω) is a compact complex manifold of the complex dimension $n \geq 3$. For any two complex forms α and β, we frequently use the following formulas: if $|\alpha| + |\beta| = 2n-1$, then

$$\int_X \alpha \wedge \partial\beta = (-1)^{|\beta|}\int_X \partial\alpha \wedge \beta = -(-1)^{|\alpha|}\int_X \partial\alpha\wedge\beta, \tag{2.1}$$

$$\int_X \alpha \wedge \overline{\partial}\beta = (-1)^{|\beta|}\int_X \overline{\partial}\alpha \wedge \beta = -(-1)^{|\alpha|}\int_X \overline{\partial}\alpha\wedge\beta. \tag{2.2}$$

Another useful formula is

$$\alpha \wedge \alpha = 0, \quad \text{if } |\alpha| \text{ is odd.} \tag{2.3}$$

By the definition of operators ∂ and $\overline{\partial}$, one has

$$\partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{2.4}$$

Hence the complex conjugate of the operator $\sqrt{-1}\partial\overline{\partial}$ is itself.

2.1. **The main idea.** Similarly in [3], we consider $\psi(s,t) = s\cdot\varphi_t$ and we can show that (see (2.49))

$$\frac{2I^0}{n(n-1)\sqrt{-1}} = \frac{I^1}{a_1} - \frac{I^2}{a_2} + c_1, \tag{2.5}$$

$$\frac{I^3}{a_3} + \frac{I^4}{a_4} = \frac{3}{-(n-2)\sqrt{-1}}c_1 + c_2, \tag{2.6}$$

where I^i are functionals which can be determined[1], c_j are also functionals but may not be determined, and a_k are nonzero constants which can be determined later. We can use equation (2.6) to eliminate the undetermined term c_1, but there arises another undetermined term c_2. Our strategy is to find a determined expression for c_2. To achieve this we construct two sequences $\{I^{2i+1}\}_{2\leq i\leq n-2}$, which can be determined, and $\{c_i\}_{2\leq i\leq n-2}$, which may not be determined, satisfying

$$\frac{I^{2i+1}}{a_{2i+1}} + \frac{I^{2i+2}}{a_{2i+2}} = \frac{i+2}{n-(i+1)}c_i + c_{i+1}, \quad 2\leq i\leq n-2, \tag{2.7}$$

where $I^{2i+2} := \overline{I^{2i+1}}$, $a_{2i+2} := \overline{a_{2i+1}}$ and a_{2i+1} can be determined later. By our construction we have $c_{n-1}\equiv 0$ which gives us the determined and explicit formula for c_2 in terms of I^{2i+1} and a_{2i+1}. More precisely, setting

$$J_i := \frac{I^{2i+1}}{a_{2i+1}} + \frac{I^{2i+2}}{a_{2i+2}}, \tag{2.8}$$

yields

$$c_{i+1} = -\frac{i+2}{n-(i+1)}c_i + J_i, \quad 2\leq i\leq n-2. \tag{2.9}$$

By induction on i we obtain

$$c_i = (-1)^{i-2}\frac{(i+1)!(n-i-1)!}{3!(n-3)!}c_2 + \sum_{k=2}^{i-1}(-1)^{i-1-k}\frac{(i+1)!(n-i-1)!}{(k+2)!(n-k-2)!}J_k. \tag{2.10}$$

Since $c_{n-1}\equiv 0$ it follows that

$$\begin{aligned} c_2 &= (-1)^{n-3}\frac{3!(n-3)!}{n!}\left[0 - \sum_{k=2}^{n-2}(-1)^{n-2-k}\frac{n!}{(k+2)!(n-k-2)!}J_k\right] \\ &= \sum_{k=2}^{n-2}(-1)^k\frac{3!(n-3)!}{(k+2)!(n-k-2)!}J_k, \quad n\geq 4. \end{aligned} \tag{2.11}$$

When X is a three-fold, we knew that $c_2 = 0$ [3], but this can be seen from (2.11) if we take $n = 3$. Hence the formula (2.11) holds for any $n\geq 3$.

2.2. **The definitions of c_1 and c_2.** Firstly, we consider the "Kähler part" of Mabuchi functional. Let

$$\mathcal{L}^0_\omega(\varphi',\varphi'') := \frac{1}{V_\omega}\int_0^1\int_X \dot\varphi_t\omega^n_{\varphi_t}dt. \tag{2.12}$$

[1] A functional $\mathcal{I}$ is said to be determined if $d\Psi = \mathcal{I}\cdot dt\wedge ds$ for some 1-form Ψ on $[0,1]\times[0,1]$.

As in [2, 3], we set $\psi(s,t) := s\cdot\varphi_t$, $0\leq t,s\leq 1$, and consider the corresponding 1-form on $[0,1]\times[0,1]$,

$$\Psi^0 = \left(\int_X \frac{\partial\psi}{\partial s}\omega_\psi^n\right)ds + \left(\int_X \frac{\partial\psi}{\partial t}\omega_\psi^n\right)dt. \tag{2.13}$$

Taking the differential on both sides implies

$$d\Psi^0 = I^0\cdot dt\wedge ds \tag{2.14}$$

where

$$I^0 := \int_X \frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial s}\omega_\psi^n\right) - \int_X \frac{\partial}{\partial s}\left(\frac{\partial\psi}{\partial t}\omega_\psi^n\right). \tag{2.15}$$

The explicit expression of I^0 can be determined as follows:

$$\begin{aligned}
I^0 &= \int_X\left[\frac{\partial\psi}{\partial s}n\omega_\psi^{n-1}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) - \frac{\partial\psi}{\partial t}n\omega_\psi^{n-1}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right]\\
&= \int_X n\frac{\partial\psi}{\partial s}\omega_\psi^{n-1}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) - \int_X n\frac{\partial\psi}{\partial t}\omega_\psi^{n-1}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\\
&= \int_X n\frac{\partial\psi}{\partial s}\omega_\psi^{n-1}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) + \int_X n\frac{\partial\psi}{\partial t}\omega_\psi^{n-1}\wedge\sqrt{-1}\overline{\partial}\partial\left(\frac{\partial\psi}{\partial s}\right)\\
&= -\int_X n\frac{\partial\psi}{\partial s}\omega_\psi^{n-1}\wedge\sqrt{-1}\overline{\partial}\partial\left(\frac{\partial\psi}{\partial t}\right) - \int_X n\frac{\partial\psi}{\partial t}\omega_\psi^{n-1}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right).
\end{aligned}$$

Here we have two slightly different expressions of I^0, and in the following we will use those expressions to find $c_1 = A_1 + B_1 - (\overline{A_1} + \overline{B_1})$, where $A_1, B_1, \overline{A_1}$ and $\overline{B_1}$ are determined later. This technique will be frequently used in many places. Hence

$$\begin{aligned}
I^0 &= \int_X -n\sqrt{-1}\partial\left(\frac{\partial\psi}{\partial s}\omega_\psi^{n-1}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) + \int_X -n\sqrt{-1}\overline{\partial}\left(\frac{\partial\psi}{\partial t}\omega_\psi^{n-1}\right)\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\\
&= \int_X -n\sqrt{-1}\left[\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-1} + (n-1)\frac{\partial\psi}{\partial s}\omega_\psi^{n-2}\wedge\partial\omega\right]\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&+ \int_X -n\sqrt{-1}\left[\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-1} + (n-1)\frac{\partial\psi}{\partial t}\omega_\psi^{n-2}\wedge\overline{\partial}\omega\right]\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\\
&= \int_X -n(n-1)\sqrt{-1}\frac{\partial\psi}{\partial s}\omega_\psi^{n-2}\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&+ \int_X -n(n-1)\sqrt{-1}\frac{\partial\psi}{\partial t}\omega_\psi^{n-2}\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial s}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
I^0 &= \int_X -n(n-1)\sqrt{-1}\frac{\partial\psi}{\partial s}\omega_\psi^{n-2}\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&+ \int_X -n(n-1)\sqrt{-1}\frac{\partial\psi}{\partial t}\omega_\psi^{n-2}\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial s}\right).
\end{aligned} \tag{2.16}$$

Using another expression of I^0, or taking the complex conjugate on both sides of (2.16) since I^0 is real, one has

$$
\begin{aligned}
I^0 &= \int_X n(n-1)\sqrt{-1}\frac{\partial\psi}{\partial s}\omega_\psi^{n-2}\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial t}\right) \\
&+ \int_X n(n-1)\sqrt{-1}\frac{\partial\psi}{\partial t}\omega_\psi^{n-2}\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right). \qquad (2.17)
\end{aligned}
$$

Hence, adding (2.17) to (2.16) and dividing by $n(n-1)\sqrt{-1}$ on both sides, we deduce

$$
\begin{aligned}
\frac{2I^0}{n(n-1)\sqrt{-1}} &= \int_X \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\frac{\partial\psi}{\partial s}\wedge\omega_\psi^{n-2}\wedge\partial\omega + \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\frac{\partial\psi}{\partial t}\wedge\omega_\psi^{n-2}\wedge\overline{\partial}\omega \\
&- \int_X \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\frac{\partial\psi}{\partial t}\wedge\omega_\psi^{n-2}\wedge\partial\omega - \int_X \partial\left(\frac{\partial\psi}{\partial t}\right)\frac{\partial\psi}{\partial s}\wedge\omega_\psi^{n-2}\wedge\overline{\partial}\omega. \qquad (2.18)
\end{aligned}
$$

According to the expression (2.18), we introduce two functionals

$$
\mathcal{L}^1_\omega(\varphi',\varphi'') := \frac{1}{V_\omega}\int_0^1\int_X a_1\partial\omega\wedge\omega_{\varphi_t}^{n-2}\wedge(\overline{\partial}\dot{\varphi}_t\cdot\varphi_t)dt, \qquad (2.19)
$$

$$
\mathcal{L}^2_\omega(\varphi',\varphi'') := \frac{1}{V_\omega}\int_0^1\int_X a_2\overline{\partial}\omega\wedge\omega_{\varphi_t}^{n-2}\wedge(\partial\dot{\varphi}_t\cdot\varphi_t)dt. \qquad (2.20)
$$

Here a_1, a_2 are non-zero constants determined later and we require $\overline{a_1} = a_2$. Satisfying this condition, a_1 and a_2 have lots of solutions. In the following we will see that we take only two special cases: a_i are purely complex numbers; a_i are real numbers. Consider the corresponding two 1-forms on $[0,1]\times[0,1]$,

$$
\begin{aligned}
\Psi^1 &= \left[\int_X a_1\partial\omega\wedge\omega_\psi^{n-2}\wedge\left(\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\cdot\psi\right)\right]ds \qquad (2.21)\\
&+ \left[\int_X a_1\partial\omega\wedge\omega_\psi^{n-2}\wedge\left(\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\cdot\psi\right)\right]dt, \\
\Psi^2 &= \left[\int_X a_2\overline{\partial}\omega\wedge\omega_\psi^{n-2}\wedge\left(\partial\left(\frac{\partial\psi}{\partial s}\right)\cdot\psi\right)\right]ds \qquad (2.22)\\
&+ \left[\int_X a_2\overline{\partial}\omega\wedge\omega_\psi^{n-2}\wedge\left(\partial\left(\frac{\partial\psi}{\partial t}\right)\cdot\psi\right)\right]dt.
\end{aligned}
$$

Firstly, we compute the differential of Ψ^1, and the differential $d\Psi^2$ can be easily written down only by taking the complex conjugate on both sides. Calculate

$$
d\Psi^1 = I^1\cdot dt\wedge ds \qquad (2.23)
$$

where

$$
\begin{aligned}
I^1 &= \int_X a_1\frac{\partial}{\partial t}\left[\partial\omega\wedge\omega_\psi^{n-2}\wedge\left(\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\cdot\psi\right)\right] \qquad (2.24)\\
&- \int_X a_1\frac{\partial}{\partial s}\left[\partial\omega\wedge\omega_\psi^{n-2}\wedge\left(\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\cdot\psi\right)\right].
\end{aligned}
$$

Dividing by a_1 on both sides of (2.24), we have

$$
\begin{aligned}
\frac{I^1}{a_1} &= \int_X -\frac{\partial}{\partial t}\left[\left(\psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right)\wedge\omega_\psi^{n-2}\wedge\partial\omega\right] \\
&+ \int_X \frac{\partial}{\partial s}\left[\left(\psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\right)\wedge\omega_\psi^{n-2}\wedge\partial\omega\right] \\
&= \int_X -\left[\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)+\psi\cdot\overline{\partial}\left(\frac{\partial^2\psi}{\partial t\partial s}\right)\right]\wedge\omega_\psi^{n-2}\wedge\partial\omega \\
&+ \int_X \left[\frac{\partial\psi}{\partial s}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)+\psi\cdot\overline{\partial}\left(\frac{\partial^2\psi}{\partial s\partial t}\right)\right]\wedge\omega_\psi^{n-2}\wedge\partial\omega \\
&+ \int_X \psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge(n-2)\omega_\psi^{n-3}\wedge-\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega \\
&+ \int_X \psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge(n-2)\omega_\psi^{n-3}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega \\
&= \int_X -\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-2}\wedge\partial\omega+\int_X \frac{\partial\psi}{\partial s}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-2}\wedge\partial\omega \\
&+ \int_X \psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge(n-2)\omega_\psi^{n-3}\wedge-\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega \\
&+ \int_X \psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge(n-2)\omega_\psi^{n-3}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega.
\end{aligned}
$$

To simplify the notation, we set

$$A_1 := \int_X \psi(n-2)\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-3}\wedge\partial\omega\wedge-\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right), \tag{2.25}$$

$$B_1 := \int_X \psi(n-2)\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-3}\wedge\partial\omega\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right). \tag{2.26}$$

Consequently, I^1/a_1 can be written as

$$
\begin{aligned}
\frac{I^1}{a_1} &= \int_X -\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-2}\wedge\partial\omega+\int_X \frac{\partial\psi}{\partial s}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-2}\wedge\partial\omega \\
&+ A_1+B_1. \qquad (2.27)
\end{aligned}
$$

Similarly, we can define I^2 by

$$d\Psi^2 = I^2\cdot dt\wedge ds, \tag{2.28}$$

where

$$
\begin{aligned}
\frac{I^2}{a_2} &:= \int_X -\frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-2}\wedge\overline{\partial}\omega+\int_X \frac{\partial\psi}{\partial s}\cdot\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-2}\wedge\overline{\partial}\omega \\
&+ \overline{A_1}+\overline{B_1}. \qquad (2.29)
\end{aligned}
$$

Consequently, from (2.18), (2.27) and (2.29),

$$\frac{2I^0}{n(n-1)\sqrt{-1}} = \frac{I^1}{a_1}-\frac{I^2}{a_2}+(A_1+B_1)-(\overline{A_1}+\overline{B_1}). \tag{2.30}$$

By a direct computation and using (2.1) and (2.3), one can show that the sum $A_1 + B_1$ has a nice form:

$$\begin{aligned}
A_1 &= \int_X \sqrt{-1}\partial\left[(n-2)\psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-3}\wedge\partial\omega\right]\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X \sqrt{-1}(n-2)\left[\partial\left(\psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right)\wedge\omega_\psi^{n-3}\wedge\partial\omega\right]\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X \sqrt{-1}(n-2)\left[\partial\psi\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)+\psi\cdot\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right]\wedge\omega_\psi^{n-3}\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X (n-2)\psi\cdot\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-3}\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&+ \int_X (n-2)\sqrt{-1}\partial\psi\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-3}\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X -(n-2)\psi\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-3}\wedge\partial\omega\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\\
&+ \int_X -(n-2)\sqrt{-1}\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-3}\\
&= -B_1 - \int_X (n-2)\sqrt{-1}\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-3}.
\end{aligned}$$

Adding the term B_1 on both sides gives

$$A_1 + B_1 = \int_X -(n-2)\sqrt{-1}\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-3}. \tag{2.31}$$

According to (2.31), we define

$$\mathcal{L}^3_\omega(\varphi',\varphi'') := \frac{1}{V_\omega}\int_0^1\int_X a_3\partial\varphi_t\wedge\partial\omega\wedge\overline{\partial}\dot{\varphi}_t\wedge\overline{\partial}\varphi_t\wedge\omega_{\varphi_t}^{n-3}, \tag{2.32}$$

$$\mathcal{L}^4_\omega(\varphi',\varphi'') := \frac{1}{V_\omega}\int_0^1\int_X a_4\overline{\partial}\varphi_t\wedge\overline{\partial}\omega\wedge\partial\dot{\varphi}_t\wedge\partial\varphi_t\wedge\omega_{\varphi_t}^{n-3}, \tag{2.33}$$

where a_3, a_4 are nonzero constants determined later and we require $\overline{a_3} = a_4$. Consider

$$\begin{aligned}
\Psi^3 &= \left[\int_X a_3\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3}\right]ds\\
&+ \left[\int_X a_3\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3}\right]dt,
\end{aligned}\tag{2.34}$$

$$\begin{aligned}
\Psi^4 &= \left[\int_X a_4\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\psi\wedge\omega_\psi^{n-3}\right]ds\\
&+ \left[\int_X a_4\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\psi\wedge\omega_\psi^{n-3}\right]dt.
\end{aligned}\tag{2.35}$$

Calculate

$$d\Psi^3 = I^3\cdot dt\wedge ds, \tag{2.36}$$

where

$$I^3 := \int_X a_3 \frac{\partial}{\partial t}\left[\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3}\right] - \int_X a_3 \frac{\partial}{\partial s}\left[\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3}\right]. \tag{2.37}$$

Also, we can calculate

$$d\Psi^4 = I^4\cdot dt\wedge ds, \tag{2.38}$$

where

$$I^4 := \int_X a_4 \frac{\partial}{\partial t}\left[\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\psi\wedge\omega_\psi^{n-3}\right] - \int_X a_4 \frac{\partial}{\partial s}\left[\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\psi\wedge\omega_\psi^{n-3}\right]. \tag{2.39}$$

Hence

$$\begin{aligned}
\frac{I^3}{a_3} &= \int_X \Bigg[\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3} \\
&+ \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial^2\psi}{\partial t\partial s}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3} + \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-3} \\
&+ \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge(n-3)\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\Bigg] \\
&- \int_X \Bigg[\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3} \\
&+ \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial^2\psi}{\partial s\partial t}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3} + \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-3} \\
&+ \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge(n-3)\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\Bigg].
\end{aligned}$$

The second term and the sixth term cancel with each other, and the third term and the seventh term are the same, so we have

$$\begin{aligned}
\frac{I^3}{a_3} &= \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3} \\
&+ \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3} \\
&+ 2\int_X \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-3} \\
&+ \int_X (n-3)\partial\psi\wedge\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&- \int_X (n-3)\partial\psi\wedge\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \\
&= H_1 + \frac{2}{-(n-2)\sqrt{-1}}(A_1+B_1) + A_2 + B_2,
\end{aligned}$$

where

$$\begin{aligned}
H_1 &:= \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3} \\
&+ \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3}
\end{aligned}\tag{2.40}$$

$$\begin{aligned}
A_2 &:= \int_X (n-3)\partial\psi\wedge\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \\
&\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)
\end{aligned}\tag{2.41}$$

$$\begin{aligned}
B_2 &:= -\int_X (n-3)\partial\psi\wedge\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right).
\end{aligned}\tag{2.42}$$

Similarly,

$$\frac{I^4}{a_4} = \overline{H_1} + \frac{2}{(n-2)\sqrt{-1}}(\overline{A_1}+\overline{B_1}) + \overline{A_2} + \overline{B_2}.\tag{2.43}$$

The hard part is to find some suitable expression of H_1. In the following we will see that $H_1 + \overline{H_1}$ has a nice form which contains only $A_1, B_1, \overline{A_1}$, and $\overline{B_1}$.

Now we compute H_1, using (2.3) and (2.4):

$$\begin{aligned}
H_1 &= \int_X \partial\left[\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3}\right]\frac{\partial\psi}{\partial t} \\
&+ \int_X \overline{\partial}\left[\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3}\right]\frac{\partial\psi}{\partial t} \\
&= \int_X \frac{\partial\psi}{\partial t}\left[\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3} - \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\left(\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3}\right)\right] \\
&+ \int_X \frac{\partial\psi}{\partial t}\left[\overline{\partial}\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3} - \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3}\right)\right] \\
&= \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\left(\partial\overline{\partial}\psi\wedge\omega_\psi^{n-3} - \overline{\partial}\psi\wedge(n-3)\omega_\psi^{n-4}\wedge\partial\omega\right) \\
&+ \int_X -\frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3} \\
&+ \int_X -\frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge(n-3)\omega_\psi^{n-4}\wedge\overline{\partial}\omega \\
&= \int_X \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\omega_\psi^{n-3}\wedge\partial\psi \\
&+ \int_X \frac{\partial\psi}{\partial t}\cdot -\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\left[\overline{\partial}\partial\omega\wedge\omega_\psi^{n-3} - \partial\omega\wedge(n-3)\omega_\psi^{n-4}\wedge\overline{\partial}\omega\right]\wedge\partial\psi \\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3} \\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge(n-3)\omega_\psi^{n-4}\wedge\overline{\partial}\omega
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
(2.44)\qquad H_1 &= \int_X \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-3}\\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\partial\psi\wedge\omega_\psi^{n-3}\\
&+ \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge(n-3)\omega_\psi^{n-4}\wedge\overline{\partial}\omega\wedge\partial\psi\\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3}\\
&+ \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge(n-3)\omega_\psi^{n-4}\wedge\overline{\partial}\omega\wedge\overline{\partial}\psi,
\end{aligned}$$

and, taking the complex conjugate yields, using (2.3) and (2.4)

$$\begin{aligned}
(2.45)\qquad \overline{H_1} &= \int_X \overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-3}\\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\overline{\partial}\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-3}\\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\overline{\partial}\omega\wedge\partial\psi\wedge\omega_\psi^{n-3}\\
&+ \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\omega\wedge(n-3)\omega_\psi^{n-4}\wedge\partial\omega\wedge\overline{\partial}\psi\\
&+ \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\omega\wedge(n-3)\omega_\psi^{n-4}\wedge\partial\omega\wedge\partial\psi.
\end{aligned}$$

Adding (2.45) to (2.45), it follows that

$$(2.46)\qquad H_1+\overline{H_1}=\frac{A_1+B_1}{-(n-2)\sqrt{-1}}+\frac{\overline{A_1}+\overline{B_1}}{(n-2)\sqrt{-1}}.$$

and, hence,

$$\begin{aligned}
\frac{I^3}{a_3}+\frac{I^4}{a_4} &= H_1+\overline{H_1}+\frac{2}{-(n-2)\sqrt{-1}}[(A_1+B_1)-(\overline{A_1}+\overline{B_1})]\\
&+ A_2+B_2+\overline{A_2}+\overline{B_2}\\
(2.47)\qquad &= \frac{3}{-(n-2)\sqrt{-1}}[(A_1+B_1)-(\overline{A_1}+\overline{B_1})]+(A_2+B_2)+(\overline{A_2}+\overline{B_2}).
\end{aligned}$$

Set

$$(2.48)\qquad c_1:=A_1+B_1-(\overline{A_1}+\overline{B_1}),\quad c_2:=A_2+B_2+\overline{A_2}+\overline{B_2}.$$

we deduce

$$(2.49)\qquad \frac{2I^0}{n(n-1)\sqrt{-1}}=\frac{I^1}{a_1}-\frac{I^2}{a_2}+c_1,\quad \frac{I^3}{a_3}+\frac{I^4}{a_4}=\frac{3}{-(n-2)\sqrt{-1}}c_1+c_2.$$

2.3. **The constructions of I^5 and I^6.** To give the general construction of I^{2i+1} and I^{2i+2}, we firstly consider some special cases. In this subsection we give the construction of I^5 and I^6, and in the next subsection the construction of I^7 and I^8. Finally, we give the general construction.

From (2.41), it follows that

$$\begin{aligned}
A_2 &= \int_X (n-3)\partial\psi\wedge\overline{\partial}\psi\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-4}\wedge\partial\omega\wedge-\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X \sqrt{-1}\partial\left[(n-3)\partial\psi\wedge\overline{\partial}\psi\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-4}\wedge\partial\omega\right]\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X \sqrt{-1}(n-3)\left[\partial\left(\partial\psi\wedge\overline{\partial}\psi\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right)\wedge\omega_\psi^{n-4}\wedge\partial\omega\right]\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X -\sqrt{-1}(n-3)\left[\partial\psi\wedge\left(\partial\overline{\partial}\psi\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)-\overline{\partial}\psi\wedge\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right)\right]\\
&\quad\wedge\omega_\psi^{n-4}\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X (n-3)\partial\psi\wedge-\sqrt{-1}\partial\overline{\partial}\psi\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-4}\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&+ \int_X (n-3)\partial\psi\wedge\overline{\partial}\psi\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-4}\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X (n-3)\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi-B_2;
\end{aligned}$$

hence, by the definition (2.42), we have

$$A_2+B_2=\int_X (n-3)\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi. \tag{2.50}$$

Motivated by (2.50), we set

$$\mathcal{L}^5_\omega(\varphi',\varphi''):=\frac{1}{V_\omega}\int_0^1\int_X a_5\partial\varphi_t\wedge\partial\omega\wedge\overline{\partial}\dot{\varphi}_t\wedge\overline{\partial}\varphi_t\wedge\omega_{\varphi_t}^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\varphi_t, \tag{2.51}$$

$$\mathcal{L}^6_\omega(\varphi',\varphi''):=\frac{1}{V_\omega}\int_0^1\int_X a_6\overline{\partial}\varphi_t\wedge\overline{\partial}\omega\wedge\partial\dot{\varphi}_t\wedge\partial\varphi_t\wedge\omega_{\varphi_t}^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\varphi_t. \tag{2.52}$$

Consider again the 1-forms

$$\begin{aligned}
\Psi^5 &= \left[\int_X a_5\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right]ds\\
&+ \left[\int_X a_5\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right]dt, \qquad (2.53)\\
\Psi^6 &= \left[\int_X a_6\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right]ds\\
&+ \left[\int_X a_6\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right]dt. \qquad (2.54)
\end{aligned}$$

The differential of Ψ^5 is given by

$$d\Psi^5=I^5\cdot dt\wedge ds, \tag{2.55}$$

where

$$
\begin{aligned}
\frac{I^5}{a_5} \quad := \quad & \int_X \frac{\partial}{\partial t}\left[\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right] \\
- \quad & \int_X \frac{\partial}{\partial s}\left[\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right].
\end{aligned} \tag{2.56}
$$

Hence, using (2.50),

$$
\begin{aligned}
\frac{I^5}{a_5} \quad = \quad & \int_X \left[\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right. \\
+ \quad & \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial^2\psi}{\partial t\partial s}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge(n-4)\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \left.\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\right] \\
- \quad & \int_X \left[\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right. \\
+ \quad & \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial^2\psi}{\partial s\partial t}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge(n-4)\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \left.\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right] \\
= \quad & \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & 2\int_X \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \int_X \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge(n-4)\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
- \quad & \int_X \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge(n-4)\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
+ \quad & \frac{A_2}{n-3}+\frac{B_2}{n-3} \\
= \quad & H_2+\frac{3}{n-3}(A_2+B_2)+A_3+B_3,
\end{aligned}
$$

where

$$\begin{aligned}
H_2 &:= \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
&+ \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi, \qquad (2.57)\\
A_3 &:= \int_X (n-4)\partial\psi\wedge\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&\quad \wedge\sqrt{-1}\partial\overline{\partial}\psi, \qquad (2.58)\\
B_3 &:= \int_X (n-4)\partial\psi\wedge\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-5}\wedge-\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \\
&\quad \wedge\sqrt{-1}\partial\overline{\partial}\psi. \qquad (2.59)
\end{aligned}$$

Similarly, for

$$d\Psi^6 =: I^6\cdot dt\wedge ds, \tag{2.60}$$

we have

$$\frac{I^6}{a_6} = \overline{H_2} + \frac{3}{n-3}(\overline{A_2}+\overline{B_2}) + \overline{A_3} + \overline{B_3}. \tag{2.61}$$

As (2.44), the hard part H_2 is calculated as follows:

$$\begin{aligned}
H_2 &= \int_X \partial\left[\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right]\frac{\partial\psi}{\partial t} \\
&+ \int_X \overline{\partial}\left[\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right]\frac{\partial\psi}{\partial t} \\
&= \int_X \frac{\partial\psi}{\partial t}\left[\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right. \\
&- \left.\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\left(\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right)\right] \\
&+ \int_X \frac{\partial\psi}{\partial t}\left[\overline{\partial}\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right. \\
&- \left.\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right)\right] \\
&= \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge(n-4)\omega_\psi^{n-5}\wedge\overline{\partial}\omega\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
&= \int_X \sqrt{-1}\partial\left[\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-4}\right]\wedge\partial\psi \\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi \\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge(n-4)\omega_\psi^{n-5}\wedge\overline{\partial}\omega\wedge\sqrt{-1}\partial\overline{\partial}\psi.
\end{aligned}$$

Hence

$$
\begin{aligned}
(2.62)\quad H_2 &= \int_X \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\psi\wedge\partial\omega\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\psi\wedge\overline{\partial}\partial\omega\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\\
&+ \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\omega\wedge(n-4)\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\psi\wedge\partial\psi\\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge\overline{\partial}\partial\omega\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\\
&+ \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\omega\wedge(n-4)\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\psi\wedge\overline{\partial}\psi
\end{aligned}
$$

and, consequently, after taking the complex conjugate on both sides,

$$
\begin{aligned}
(2.63)\quad \overline{H_2} &= \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge\partial\overline{\partial}\omega\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\\
&+ \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\omega\wedge\partial\omega\wedge(n-4)\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\psi\wedge\overline{\partial}\psi\\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\psi\wedge\partial\overline{\partial}\omega\wedge\omega_\psi^{n-4}\wedge\sqrt{-1}\partial\overline{\partial}\psi\\
&+ \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\omega\wedge\partial\omega\wedge(n-4)\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\psi\wedge\partial\psi.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(2.64)\quad H_2+\overline{H_2} &= \frac{A_2+B_2}{n-3}+\frac{\overline{A_2}+\overline{B_2}}{n-3},\\
(2.65)\quad \frac{I^5}{a_5}+\frac{I^6}{a_6} &= \frac{4}{n-3}(A_2+B_2+\overline{A_2}+\overline{B_2})+(A_3+B_3+\overline{A_3}+\overline{B_3}).
\end{aligned}
$$

If we set

$$
(2.66)\qquad c_3 := A_3+B_3+\overline{A_3}+\overline{B_3}
$$

we can rewrite (2.64) and (2.65) as

$$
(2.67)\qquad H_2+\overline{H_2}=\frac{c_2}{n-3},\quad \frac{I^5}{a_5}+\frac{I^6}{a_6}=\frac{4}{n-3}c_2+c_3.
$$

2.4. **The constructions of I^7 and I^8.** Recall

$$
A_3=\int_X\left[(n-4)\partial\psi\wedge\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right]\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right).
$$

By a direct computation, one deduces that

$$\begin{aligned}
A_3 &= \int_X -\sqrt{-1}\partial\left[(n-4)\partial\psi\wedge\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right]\\
&\quad \wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X \sqrt{-1}(n-4)\partial\psi\wedge\partial\left(\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right)\\
&\quad \wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X \sqrt{-1}(n-4)\partial\psi\wedge\left[\partial\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right.\\
&+ \left.\overline{\partial}\psi\wedge\partial\omega\wedge\partial\left(\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right)\right]\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X \sqrt{-1}(n-4)\partial\psi\wedge\left[\partial\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right.\\
&+ \left.\overline{\partial}\psi\wedge\partial\omega\wedge\left(\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right)\right]\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&= \int_X (n-4)\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2\\
&+ \int_X (n-4)\partial\psi\wedge\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\psi\\
&= -B_3+\int_X (n-4)\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2.
\end{aligned}$$

Consequently, it follows that

$$A_3+B_3=\int_X (n-4)\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2. \tag{2.68}$$

Now we introduce the corresponding functionals

$$\mathcal{L}^7_\omega(\varphi',\varphi'') := \frac{1}{V_\omega}\int_0^1\int_X a_7\partial\varphi_t\wedge\partial\omega\wedge\overline{\partial}\dot{\varphi}_t\wedge\overline{\partial}\varphi_t\wedge\omega_{\varphi_t}^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\varphi_t)^2, \tag{2.69}$$

$$\mathcal{L}^8_\omega(\varphi',\varphi'') := \frac{1}{V_\omega}\int_0^1\int_X a_8\overline{\partial}\varphi_t\wedge\overline{\partial}\omega\wedge\partial\dot{\varphi}_t\wedge\partial\varphi_t\wedge\omega_{\varphi_t}^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\varphi_t)^2 \tag{2.70}$$

and consider the 1-forms

$$\begin{aligned}
\Psi^7 &= \left[\int_X a_7\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2\right]ds\\
&+ \left[\int_X a_7\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2\right]dt, \qquad (2.71)\\
\Psi^8 &= \left[\int_X a_8\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\psi\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2\right]ds\\
&+ \left[\int_X a_8\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\psi\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2\right]dt. \qquad (2.72)
\end{aligned}$$

So, we have the expression,

$$d\Psi^7 = I^7 \cdot dt \wedge ds, \quad d\Psi^8 = I^8 \cdot dt \wedge ds, \tag{2.73}$$

where

$$\begin{aligned}
\frac{I^7}{a_7} &:= \int_X \frac{\partial}{\partial t}\left[\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2\right] \\
&- \int_X \frac{\partial}{\partial s}\left[\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2\right] \\
&= \int_X \left[\partial\left(\frac{\partial\psi}{\partial t}\right) \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2\right. \\
&+ \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial^2\psi}{\partial t\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
&+ \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
&+ \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge (n-5)\omega_\psi^{n-6} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
&+ \left.\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge 2\sqrt{-1}\partial\overline{\partial}\psi \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\right] \\
&- \int_X \left[\partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2\right. \\
&+ \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial^2\psi}{\partial s\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
&+ \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
&+ \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial\psi} \wedge (n-5)\omega_\psi^{n-6} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
&+ \left.\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge 2\sqrt{-1}\partial\overline{\partial}\psi \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right] \\
&= \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
&+ \int_X \partial\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \partial\omega \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\partial\overline{\partial}\psi)^2 \\
&+ 2\int_X \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-5} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
&+ \int_X \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge (n-5)\omega_\psi^{n-6} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
&- \int_X \partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge (n-5)\omega_\psi^{n-6} \wedge \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
&+ \frac{2}{n-4}(A_3 + B_3) \\
&= H_3 + \frac{4}{n-4}(A_3 + B_3) + A_4 + B_4
\end{aligned}$$

where

$$
\begin{aligned}
H_3 &:= \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
&+ \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2 \qquad (2.74)
\end{aligned}
$$

and

$$
\begin{aligned}
A_4 &:= \int_X (n-5)\partial\psi\wedge\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-6} \\
&\quad \wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2, \qquad (2.75) \\
B_4 &:= \int_X (n-5)\partial\psi\wedge\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-6} \\
&\quad \wedge-\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2. \qquad (2.76)
\end{aligned}
$$

Hence

$$
\frac{I^7}{a_7} = H_3 + \frac{4}{n-4}(A_3+B_3) + A_4 + B_4. \qquad (2.77)
$$

Similarly

$$
\frac{I^8}{a_8} = \overline{H_3} + \frac{4}{n-4}(\overline{A_3}+\overline{B_3}) + \overline{A_4} + \overline{B_4}. \qquad (2.78)
$$

Calculate

$$
\begin{aligned}
H_3 &= \int_X \partial\left[\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2\right]\frac{\partial\psi}{\partial t} \\
&+ \int_X \overline{\partial}\left[\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2\right]\frac{\partial\psi}{\partial t} \\
&= \int_X \frac{\partial\psi}{\partial t}\left[\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2\right. \\
&- \left.\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\left(\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-5}\wedge\left(\sqrt{-1}\partial\overline{\partial}\psi\right)^2)\right]\right. \\
&+ \int_X \frac{\partial\psi}{\partial t}\left[\overline{\partial}\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2\right. \\
&- \left.\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2\right)\right] \\
&= \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2 \\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge(n-5)\omega_\psi^{n-6}\wedge\overline{\partial}\omega\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2.
\end{aligned}
$$

Since

$$\int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2 \tag{2.79}$$

$$\begin{aligned}
&= \int_X\left[\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\psi\right]\wedge-\sqrt{-1}\overline{\partial}\partial\psi\\
&= \int_X\sqrt{-1}\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\psi\wedge\partial\psi\\
&- \int_X\sqrt{-1}\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\Big(\overline{\partial}\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-5}\wedge\sqrt{-1}\partial\overline{\partial}\psi\\
&- \partial\omega\wedge\partial\overline{\partial}\psi\wedge(n-5)\omega_\psi^{n-6}\wedge\overline{\partial}\omega\wedge\sqrt{-1}\partial\overline{\partial}\psi\Big)\wedge\partial\psi,
\end{aligned}$$

it follows that

$$\begin{aligned}
H_3 &= \int_X\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\psi\wedge\partial\omega\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2\\
&- \int_X\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\psi\wedge\overline{\partial}\partial\omega\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2 \qquad (2.80)\\
&+ \int_X\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\omega\wedge(n-5)\omega_\psi^{n-6}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2\wedge\partial\psi\\
&- \int_X\frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge\overline{\partial}\partial\omega\wedge\omega_\psi^{n-5}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2\\
&+ \int_X\frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\omega\wedge(n-5)\omega_\psi^{n-6}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^2\wedge\overline{\partial}\psi.
\end{aligned}$$

Hence

$$H_3+\overline{H_3} = \frac{A_3+B_3}{n-4}+\frac{\overline{A_3}+\overline{B_3}}{n-4},, \tag{2.81}$$

$$\frac{I^7}{a_7}+\frac{I^8}{a_8} = \frac{5}{n-4}(A_3+B_3+\overline{A_3}+\overline{B_3})+(A_4+B_4+\overline{A_4}+\overline{B_4}). \tag{2.82}$$

Set

$$c_4 := A_4+B_4+\overline{A_4}+\overline{B_4}. \tag{2.83}$$

Then

$$\frac{I^7}{a_7}+\frac{I^8}{a_8}=\frac{5}{n-4}c_3+c_4. \tag{2.84}$$

2.5. **Recursion formula.** Suppose now $n\geq 4$. we define, for $2\leq i\leq n-2$,

$$\begin{aligned}
A_i &:= \int_X\left[(n-i-1)\partial\psi\wedge\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\right]\\
&\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right), \qquad (2.85)\\
B_i &;= \int_X\left[(n-i-1)\partial\psi\wedge\overline{\partial}\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\right]\\
&\wedge-\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right). \qquad (2.86)
\end{aligned}$$

So

$$
\begin{aligned}
A_i &= \int_X \left[(n-i-1)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\right] \\
&\quad \wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X -\sqrt{-1}\partial\left[(n-i-1)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\right] \\
&\quad \wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X \sqrt{-1}(n-i-1)\partial\psi \wedge \left[\partial\overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\right. \\
&- \left.\overline{\partial}\psi \wedge \partial\left(\partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\right)\right] \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X \sqrt{-1}(n-i-1)\partial\psi \wedge \partial\overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2} \\
&\quad \wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&+ \int_X \sqrt{-1}(n-i-1)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \partial\left(\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\right) \\
&\quad \wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \\
&= \int_X (n-i-1)\partial\psi \wedge \overline{\partial}\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-2} \\
&\quad \wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \\
&+ \int_X (n-i-1)\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \omega^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1};
\end{aligned}
$$

thus

$$
A_i+B_i = \int_X (n-i-1)\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}. \tag{2.87}
$$

Define, where a_{2i+1} and a_{2i+2} are nonzero constants and we require $\overline{a_{2i+1}} = a_{2i+2}$,

$$
\begin{aligned}
\mathcal{L}_\omega^{2i+1}(\varphi',\varphi'') &:= \frac{1}{V_\omega}\int_0^1\int_X a_{2i+1}\partial\varphi_t \wedge \partial\omega \wedge \overline{\partial}\dot{\varphi}_t \wedge \overline{\partial}\varphi_t \wedge \omega_{\varphi_t}^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\varphi_t)^{i-1}, \\
\mathcal{L}_\omega^{2i+2}(\varphi',\varphi'') &:= \frac{1}{V_\omega}\int_0^1\int_X a_{2i+2}\overline{\partial}\varphi_t \wedge \overline{\partial}\omega \wedge \partial\dot{\varphi}_t \wedge \partial\varphi_t \wedge \omega_{\varphi_t}^{n-i-2} \\
&\quad \wedge(\sqrt{-1}\partial\overline{\partial}\varphi_t)^{i-1}.
\end{aligned} \tag{2.88}
$$

Consider

$$
\begin{aligned}
\Psi^{2i+1} &= \left[\int_X a_{2i+1}\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right] ds \\
&+ \left[\int_X a_{2i+1}\partial\psi \wedge \partial\omega \wedge \overline{\partial}\left(\frac{\partial\psi}{\partial t}\right) \wedge \overline{\partial}\psi \wedge \omega_\psi^{n-i-2} \wedge (\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right] dt,
\end{aligned} \tag{2.89}
$$

and

$$\begin{aligned}\Psi^{2i+2} &= \left[\int_X a_{2i+2}\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right]ds\\
&+ \left[\int_X a_{2i+2}\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right]dt. \qquad (2.90)\end{aligned}$$

So

$$d\Psi^{2i+1}=I^{2i+1}\cdot dt\wedge ds,\quad d\Psi^{2i+2}=I^{2i+2}\cdot dt\wedge ds, \qquad (2.91)$$

where

$$\begin{aligned}\frac{I^{2i+1}}{a_{2i+1}} &:= \int_X\frac{\partial}{\partial t}\left[\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right]\\
&- \int_X\frac{\partial}{\partial s}\left[\partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right]\\
&= \int_X\left[\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right.\\
&+ \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial^2\psi}{\partial t\partial s}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&+ \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&+ \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge(n-i-2)\omega_\psi^{n-i-3}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&\quad\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&+ \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(i-1)(\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\\
&\quad\left.\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\right]\\
&- \int_X\left[\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right.\\
&+ \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial^2\psi}{\partial s\partial t}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&+ \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&+ \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge(n-i-2)\omega_\psi^{n-i-3}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\\
&\quad\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&+ \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(i-1)(\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\\
&\quad\left.\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\right].\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{I^{2i+1}}{a_{2i+1}} &= \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&+ \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&+ 2\int_X \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&+ \int_X \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge(n-i-2)\omega_\psi^{n-i-3}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\\
&\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&- \int_X \partial\psi\wedge\partial\omega\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge(n-i-2)\omega_\psi^{n-i-3}\wedge\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\\
&\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}+(i-1)(A_i+B_i)/(n-i-1)\\
&= H_i+\frac{i+1}{n-(i+1)}(A_i+B_i)+A_{i+1}+B_{i+1}. \qquad (2.92)
\end{aligned}$$

Here

$$\begin{aligned}
H_i &:= \int_X -\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&+ \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}. \qquad (2.93)
\end{aligned}$$

Similarly

$$\frac{I^{2i+2}}{a_{2i+2}}=\overline{H_i}+\frac{i+1}{n-(i+1)}(\overline{A_i}+\overline{B_i})+\overline{A_{i+1}}+\overline{B_{i+1}}. \qquad (2.94)$$

Calculate

$$\begin{aligned}
H_i &= \int_X \partial\left[\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right]\frac{\partial\psi}{\partial t}\\
&+ \int_X \overline{\partial}\left[\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right]\frac{\partial\psi}{\partial t}\\
&= \int_X \frac{\partial\psi}{\partial t}\left[\partial\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right.\\
&- \left.\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\left(\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right)\right]\\
&+ \int_X \frac{\partial\psi}{\partial t}\left[\overline{\partial}\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right.\\
&- \left.\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\right)\right]\\
&= \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\\
&- \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\psi\wedge(n-i-2)\omega_\psi^{n-i-3}\wedge\overline{\partial}\omega\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}.
\end{aligned}$$

Since

$$
\begin{aligned}
& \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \\
= & \int_X \left[\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\right]\wedge(-\sqrt{-1}\overline{\partial}\partial\psi) \\
= & \int_X \sqrt{-1}\overline{\partial}\left[\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\right]\wedge\partial\psi \\
= & \int_X \sqrt{-1}\left[\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\right. \\
- & \left.\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\right)\right]\wedge\partial\psi \\
= & \int_X \sqrt{-1}\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\wedge\partial\psi \\
- & \int_X \sqrt{-1}\frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\left(\overline{\partial}\partial\omega\wedge\partial\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\right. \\
- & \left.\partial\omega\wedge\partial\overline{\partial}\psi\wedge(n-i-2)\omega_\psi^{n-i-3}\wedge\overline{\partial}\omega\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-2}\right)\wedge\partial\psi,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
H_i = & \int_X \overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\psi\wedge\partial\omega\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \\
- & \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\psi\wedge\overline{\partial}\partial\omega\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \\
+ & \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\omega\wedge(n-i-2)\omega_\psi^{n-i-3}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\wedge\partial\psi \\
- & \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\partial\omega\wedge\overline{\partial}\psi\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \\
+ & \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\omega\wedge\overline{\partial}\omega\wedge(n-i-2)\omega_\psi^{n-i-3}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\wedge\overline{\partial}\psi,
\end{aligned}
\tag{2.95}
$$

and, similarly,

$$
\begin{aligned}
\overline{H_i} = & \int_X \partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\overline{\partial}\psi\wedge\overline{\partial}\omega\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \\
- & \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\psi\wedge\partial\overline{\partial}\omega\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \\
+ & \int_X \frac{\partial\psi}{\partial t}\cdot\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\omega\wedge\partial\omega\wedge(n-i-2)\omega_\psi^{n-i-3}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\wedge\overline{\partial}\psi \\
- & \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\partial\psi\wedge\partial\overline{\partial}\omega\wedge\omega_\psi^{n-i-2}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1} \\
+ & \int_X \frac{\partial\psi}{\partial t}\cdot\overline{\partial}\left(\frac{\partial\psi}{\partial s}\right)\wedge\overline{\partial}\omega\wedge\partial\omega\wedge(n-i-2)\omega_\psi^{n-i-3}\wedge(\sqrt{-1}\partial\overline{\partial}\psi)^{i-1}\wedge\partial\psi.
\end{aligned}
\tag{2.96}
$$

So, for $2 \le i \le n-2$,

$$
\begin{aligned}
H_i + \overline{H_i} &= \frac{A_i + B_i}{n-i-1} + \frac{\overline{A_i} + \overline{B_i}}{n-i-1}, \qquad (2.97)\\
\frac{I^{2i+1}}{a_{2i+1}} + \frac{I^{2i+2}}{a_{2i+2}} &= H_i + \overline{H_i} + \frac{i+1}{n-(i+1)}(A_i + B_i + \overline{A_i} + \overline{B_i})\\
&+ (A_{i+1} + B_{i+1} + \overline{A_{i+1}} + \overline{B_{i+1}})\\
&= \frac{i+2}{n-(i+1)}(A_i + B_i + \overline{A_i} + \overline{B_i})\\
&+ (A_{i+1} + B_{i+1} + \overline{A_{i+1}} + \overline{B_{i+1}}). \qquad (2.98)
\end{aligned}
$$

Recall, see (2.49),

$$
\begin{aligned}
\frac{I^3}{a_3} + \frac{I^4}{a_4} &= \frac{3\sqrt{-1}}{n-2}\left((A_1 + B_1) - (\overline{A_1} + \overline{B_1})\right) + (A_2 + B_2 + \overline{A_2} + \overline{B_2})\\
\frac{2I^0}{n(n-1)\sqrt{-1}} &= \frac{I^1}{a_1} - \frac{I^2}{a_2} + (A_1 + B_1) - (\overline{A_1} + \overline{B_1}).
\end{aligned}
$$

Let

$$
c_i := A_i + B_i + \overline{A_i} + \overline{B_i}, \quad 2 \le i \le n-1, \qquad (2.99)
$$

$$
c_1 := A_1 + B_1 - (\overline{A_1} + \overline{B_1}). \qquad (2.100)
$$

Notice that $c_{n-1} = 0$. So

$$
\frac{I^{2i+1}}{a_{2i+1}} + \frac{I^{2i+2}}{a_{2i+2}} = \frac{i+2}{n-(i+1)}c_i + c_{i+1}, \quad 2 \le i \le n-2, \qquad (2.101)
$$

$$
\frac{I^3}{a_3} + \frac{I^4}{a_4} = \frac{3}{-(n-2)\sqrt{-1}}c_1 + c_2, \qquad (2.102)
$$

and

$$
\begin{aligned}
\frac{2I^0}{n(n-1)\sqrt{-1}} &= \frac{I^1}{a_1} - \frac{I^2}{a_2} + c_1\\
&= \frac{I^1}{a_1} - \frac{I^2}{a_2} + \left(\frac{I^3}{a_3} + \frac{I^4}{a_4} - c_2\right)\frac{n-2}{3\sqrt{-1}}\\
&= \left(\frac{I^1}{a_1} - \frac{I^2}{a_2}\right) + \frac{n-2}{3\sqrt{-1}}\left(\frac{I^3}{a_3} + \frac{I^4}{a_4}\right) - \frac{n-2}{3\sqrt{-1}}c_2.
\end{aligned}
$$

It is sufficient to determine c_2. Let

$$
J_i := \frac{I^{2i+1}}{a_{2i+1}} + \frac{I^{2i+2}}{a_{2i+2}}. \qquad (2.103)
$$

Then

$$
c_{i+1} = -\frac{i+2}{n-(i+1)}c_i + J_i, \quad 2 \le i \le n-2. \qquad (2.104)
$$

To completely determine c_2, it is sufficient to solve (2.104). A direct calculation shows

$$\begin{aligned}
c_3 &= -\frac{4}{n-3}c_2 + J_2, \\
c_4 &= -\frac{5}{n-4}c_3 + J_3 \quad = \quad -\frac{5}{n-4}\left(-\frac{4}{n-3}c_2 + J_2\right) + J_3 \\
&= (-1)^2\frac{5\times 4}{(n-4)(n-3)}c_2 - \frac{5}{n-4}J_2 + J_3, \\
c_5 &= -\frac{6}{n-5}c_4 + J_4 \\
&= -\frac{6}{n-5}\left[(-1)^2\frac{5\times 4}{(n-4)(n-3)}c_2 - \frac{5}{n-4}J_2 + J_3\right] + J_4 \\
&= (-1)^3\frac{6\times 5\times 4}{(n-5)(n-4)(n-3)}c_2 + (-1)^2\frac{6\times 5}{(n-5)(n-4)}J_2 - \frac{6}{n-5}J_3 + J_4
\end{aligned}$$

Hence, we have

$$c_i = (-1)^{i-2}\frac{(i+1)!(n-i-1)!}{3!(n-3)!}c_2 + \sum_{k=2}^{i-1}(-1)^{i-1-k}\frac{(i+1)!(n-i-1)!}{(k+2)!(n-k-2)!}J_k. \tag{2.105}$$

By induction on i, we have

$$\begin{aligned}
c_{i+1} &= -\frac{i+2}{n-(i+1)}c_i + J_i \\
&= -\frac{i+2}{n-(i+1)}\left[(-1)^{i-2}\frac{(i+1)!(n-i-1)!}{3!(n-3)!}c_2\right. \\
&+ \left.\sum_{k=2}^{i-1}(-1)^{i-1-k}\frac{(i+1)!(n-i-1)!}{(k+2)!(n-k-2)!}J_k\right] + J_i \\
&= (-1)^{i+1-2}\frac{(i+2)!(n-i-2)!}{3!(n-3)!}c_2 \\
&+ \sum_{k=2}^{i-1}(-1)^{i-k}\frac{(i+2)!(n-i-2)!}{(k+2)!(n-k-2)!}J_k + J_i.
\end{aligned}$$

So (2.105) holds for $2 \le i \le n-2$ and we have

$$\begin{aligned}
c_2 &= (-1)^{i-2}\frac{3!(n-3)!}{(i+1)!(n-i-1)!} \\
&\cdot \left[c_i - \sum_{k=2}^{i-1}(-1)^{i-1-k}\frac{(i+1)!(n-i-1)!}{(k+2)!(n-k-2)!}J_k\right].
\end{aligned} \tag{2.106}$$

Setting $i = n-1$ yields

$$\begin{aligned}
c_2 &= (-1)^{n-3}\frac{3!(n-3)!}{n!} \\
&\cdot \left[c_{n-1} - \sum_{k=2}^{n-2}(-1)^{n-2-k}\frac{n!}{(k+2)!(n-k-2)!}J_k\right], \quad n \ge 3.
\end{aligned} \tag{2.107}$$

We deduce from (2.49) and (2.107) that

$$
\begin{aligned}
\frac{2I^0}{n(n-1)\sqrt{-1}} &= \left(\frac{I^0}{a_1}-\frac{I^2}{a_2}\right)+\frac{n-2}{3\sqrt{-1}}\left(\frac{I^3}{a_3}+\frac{I^4}{a_4}\right)\\
&- \frac{n-2}{3\sqrt{-1}}(-1)^{n-3}\frac{3!(n-3)!}{n!}\\
&\cdot \left[c_{n-1}-\sum_{k=2}^{n-2}(-1)^{n-2-k}\frac{n!}{(k+2)!(n-k-2)!}J_k\right]\\
&= \left(\frac{I^0}{a_1}-\frac{I^2}{a_2}\right)+\frac{n-2}{3\sqrt{-1}}\left(\frac{I^3}{a_3}+\frac{I^4}{a_4}\right)\\
&+ \frac{2(n-2)!}{\sqrt{-1}}\sum_{k=2}^{n-2}\frac{(-1)^{k+1}}{(k+2)!(n-k-2)!}\left(\frac{I^{2k+1}}{a_{2k+1}}+\frac{I^{2k+2}}{a_{2k+2}}\right).
\end{aligned}
$$

Equivalently,

$$
\frac{2I^0}{n(n-1)\sqrt{-1}}=\left(\frac{I^1}{a_1}-\frac{I^2}{a_2}\right)+\sqrt{-1}\sum_{k=1}^{n-2}(-1)^k\frac{\binom{n}{k+2}}{\binom{n}{2}}\left(\frac{I^{2k+1}}{a_{2k+1}}+\frac{I^{2k+2}}{a_{2k+2}}\right). \tag{2.108}
$$

Set

$$
\frac{1}{a_1} = -\frac{2}{n(n-1)\sqrt{-1}}, \quad \frac{1}{a_2} = \frac{2}{n(n-1)\sqrt{-1}}, \tag{2.109}
$$

$$
\frac{\sqrt{-1}(-1)^k\binom{n}{k+2}}{\binom{n}{2}a_{2k+1}} = -\frac{2}{n(n-1)\sqrt{-1}}, \quad a_{2i+1} = a_{2i+2}, \tag{2.110}
$$

we obtain

$$
\begin{aligned}
a_1 &= -\frac{n(n-1)\sqrt{-1}}{2}, \quad a_2 = \frac{n(n-1)\sqrt{-1}}{2}, && (2.111)\\
a_{2k+1} &= a_{2k+2}\\
&= \frac{\sqrt{-1}(-1)^k\binom{n}{k+2}n(n-1)\sqrt{-1}}{-2\binom{n}{2}}\\
&= \frac{(-1)^k n(n-1)\binom{n}{k+2}}{2\binom{n}{2}} = (-1)^k\binom{n}{k+2}. && (2.112)
\end{aligned}
$$

Consequently,

$$
I^0+\sum_{k=0}^{n-2}\left(I^{2k+1}+I^{2k+2}\right)=0. \tag{2.113}
$$

Theorem 2.1. *The functional, for $n \geq 2$,*

$$
\begin{aligned}
\mathcal{L}_\omega^{\mathrm{M}}(\varphi',\varphi'') \quad :=& \quad \frac{1}{V_\omega}\int_0^1\int_X \dot{\varphi}_t\omega_{\varphi_t}^n dt \\
- & \quad \frac{1}{V_\omega}\int_0^1\int_X \frac{n(n-1)\sqrt{-1}}{2}\partial\omega\wedge\omega_{\varphi_t}^{n-2}\wedge(\overline{\partial}\dot{\varphi}_t\cdot\varphi_t)dt \\
+ & \quad \frac{1}{V_\omega}\int_0^1\int_X \frac{n(n-1)\sqrt{-1}}{2}\overline{\partial}\omega\wedge\omega_{\varphi_t}^{n-2}\wedge(\partial\dot{\varphi}_t\cdot\varphi_t)dt \\
+ & \quad \sum_{i=1}^{n-2}\frac{1}{V_\omega}\int_0^1\int_X(-1)^i\binom{n}{i+2}\partial\varphi_t\wedge\partial\omega\wedge\overline{\partial}\dot{\varphi}_t\wedge\overline{\partial}\varphi_t\wedge\omega_{\varphi_t}^{n-i-2} \\
& \quad \wedge(\sqrt{-1}\partial\overline{\partial}\varphi_t)^{i-1} \\
+ & \quad \sum_{i=1}^{n-2}\frac{1}{V_\omega}\int_0^1\int_X(-1)^i\binom{n}{i+2}\overline{\partial}\varphi_t\wedge\overline{\partial}\omega\wedge\partial\dot{\varphi}_t\wedge\partial\varphi_t\wedge\omega_{\varphi_t}^{n-i-2} \\
& \quad \wedge(\sqrt{-1}\partial\overline{\partial}\varphi_t)^{i-1}.
\end{aligned}
\tag{2.114}
$$

is independent of the choice of the smooth path $\{\varphi_t\}_{0\leq t\leq 1}$ in $\mathcal{P}_\omega$ from φ' to φ''.

Proof. Using (2.113), we can prove Theorem 2.1 in the similar way as [2]. $\square$

Corollary 2.2. *Suppose $n \geq 2$. For any $\varphi \in \mathcal{P}_\omega$ one has*

$$
\begin{aligned}
\mathcal{L}_\omega^{\mathrm{M}}(\varphi) \quad := \quad \mathcal{L}_\omega^{\mathrm{M}}(0,\varphi) \quad =& \quad \frac{1}{V_\omega}\sum_{i=0}^{n}\int_X\frac{1}{n+1}\varphi\omega_\varphi^i\wedge\omega^{n-i} \\
- & \quad \sum_{i=0}^{n-2}\frac{i+1}{2V_\omega}\int_X\varphi\omega_\varphi^i\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi \\
+ & \quad \sum_{i=0}^{n-2}\frac{i+1}{2V_\omega}\int_X\varphi\omega_\varphi^i\wedge\omega^{n-2-i}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi.
\end{aligned}
\tag{2.115}
$$

Proof. Since $\mathcal{L}_\omega^{\mathrm{M}}(\varphi)$ is independent of the choice of smooth path, we pick $\varphi_t = t\varphi$, $0 \leq t \leq 1$. Then $\overline{\partial}\dot{\varphi}_t\wedge\overline{\partial}\varphi_t = \partial\dot{\varphi}_t\wedge\partial\varphi_t = 0$, and

$$
\begin{aligned}
\mathcal{L}_\omega^{\mathrm{M}}(\varphi) \quad =& \quad \frac{1}{V_\omega}\int_0^1\int_X\varphi\omega_{t\varphi}^n dt \\
- & \quad \frac{n(n-1)}{2V_\omega}\int_0^1\int_X\sqrt{-1}\partial\omega\wedge\omega_{t\varphi}^{n-2}\wedge(\overline{\partial}\varphi\cdot t\varphi)dt \\
+ & \quad \frac{n(n-1)}{2V_\omega}\int_0^1\int_X\sqrt{-1}\overline{\partial}\omega\wedge\omega_{t\varphi}^{n-2}\wedge(\partial\varphi\cdot t\varphi)dt \quad =: \quad J_0+J_1+J_2.
\end{aligned}
$$

Now we compute J_0, J_1, J_2, respectively. Using

$$\omega_{t\varphi} = \omega + t\sqrt{-1}\partial\overline{\partial}\varphi = \omega + t(\omega_\varphi - \omega) = t\omega_\varphi + (1-t)\omega,$$

it follows that

$$\begin{aligned}
J_0 &= \frac{1}{V_\omega}\int_X\int_0^1 \varphi\sum_{i=0}^n \binom{n}{i}\omega_\varphi^i\wedge\omega^{n-i}t^i(1-t)^{n-i}dt \\
&= \sum_{i=0}^n \frac{1}{V_\omega}\int_X \varphi\binom{n}{i}\omega_\varphi^i\wedge\omega^{n-i}\cdot\int_0^1 t^i(1-t)^{n-i}dt \\
&= \sum_{i=0}^n \frac{1}{V_\omega}\int_X \binom{n}{i}\frac{\Gamma(i+1)\Gamma(n-i+1)}{\Gamma(n+2)}\varphi\omega_\varphi^i\wedge\omega^{n-i} \\
&= \sum_{i=0}^n \frac{1}{V_\omega}\int_X \frac{n!}{i!(n-i)!}\frac{i!(n-i)!}{(n+1)!}\varphi\omega_\varphi^i\wedge\omega^{n-i} = \sum_{i=0}^n \frac{1}{V_\omega}\int_X \frac{1}{n+1}\varphi\omega_\varphi^i\wedge\omega^{n-i},
\end{aligned}$$

where $\Gamma(x)$ is the Gamma function. Similarly, we have

$$\begin{aligned}
J_1 &= -\frac{n(n-1)}{2V_\omega}\int_X\int_0^1 \sqrt{-1}\partial\omega\wedge[t\omega_\varphi+(1-t)\omega]^{n-2}\wedge(\overline{\partial}\varphi\cdot t\varphi)dt \\
&= -\frac{n(n-1)}{2V_\omega}\int_X\int_0^1 \sqrt{-1}\partial\omega \\
&\quad \wedge\sum_{i=0}^{n-2}\binom{n-2}{i}t^{i+1}\omega_\varphi^i(1-t)^{n-2-i}\omega^{n-2-i}\wedge(\overline{\partial}\varphi\cdot\varphi)dt \\
&= -\frac{n(n-1)}{2V_\omega}\int_X \sqrt{-1}\partial\omega\wedge\sum_{i=0}^{n-2}\omega_\varphi^i\wedge\omega^{n-2-i}\wedge(\overline{\partial}\varphi\cdot\varphi) \\
&\cdot \int_0^1\binom{n-2}{i}t^{i+1}(1-t)^{n-2-i}dt \\
&= -\frac{n(n-1)}{2V_\omega}\int_X \sqrt{-1}\partial\omega\wedge\sum_{i=0}^{n-2}\omega_\varphi^i\wedge\omega^{n-2-i}\wedge(\overline{\partial}\varphi\cdot\varphi)\cdot\frac{i+1}{n(n-1)} \\
&= \sum_{i=0}^{n-2}\frac{-1}{2V_\omega}\int_X(i+1)\varphi\omega_\varphi^i\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi.
\end{aligned}$$

Taking the complex conjugate gives

$$J_2 = \sum_{i=0}^{n-2}\frac{1}{2V_\omega}\int_X(i+1)\varphi\omega_\varphi^i\wedge\omega^{n-2-i}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi.$$

Together with the expressions of J_0, J_1 and J_2, we complete the proof. □

Remark 2.3. *When (X,ω) is a compact Kähler manifold, the functional (2.114) or (2.115) coincides with the original one.*

Let S be a non-empty set and A an additive group. A mapping $\mathcal{N}: S\times S\to A$ is said to satisfy the 1-cocycle condition if

(i) $\mathcal{N}(\sigma_1,\sigma_2)+\mathcal{N}(\sigma_2,\sigma_1)=0$;
(ii) $\mathcal{N}(\sigma_1,\sigma_2)+\mathcal{N}(\sigma_2,\sigma_3)+\mathcal{N}(\sigma_3,\sigma_1)=0$.

Corollary 2.4. *(1) The functional* $\mathcal{L}^{\mathrm{M}}_\omega$ *satisfies the* 1*-cocycle condition.*
(2) For any $\varphi \in \mathcal{P}_\omega$ *and any constant* $C \in \mathbb{R}$*, we have*

$$\mathcal{L}^{\mathrm{M}}_\omega(\varphi, \varphi + C) = C \cdot \left(1 + \frac{\mathrm{Err}_\omega(\varphi)}{V_\omega}\right), \quad \mathrm{Err}_\omega(\varphi) := \int_X \omega^n - \int_X \omega^n_\varphi. \tag{2.116}$$

In particular, if $\partial\overline{\partial}\omega = \partial\omega \wedge \overline{\partial}\omega = 0$*, then* $\mathcal{L}^{\mathrm{M}}_\omega(\varphi, \varphi + C) = C$*.*
(3) For any $\varphi_1, \varphi_2 \in \mathcal{P}_\omega$ *and any constant* $C \in \mathbb{R}$*, we have*

$$\mathcal{L}^{\mathrm{M}}_\omega(\varphi_1, \varphi_2 + C) = \mathcal{L}^{\mathrm{M}}_\omega(\varphi_1, \varphi_2) + C \cdot \left(1 - \frac{\mathrm{Err}_\omega(\varphi_2)}{V_\omega}\right). \tag{2.117}$$

In particular, if $\partial\overline{\partial}\omega = \partial\omega \wedge \overline{\partial}\omega = 0$*, then* $\mathcal{L}^{\mathrm{M}}_\omega(\varphi_1, \varphi_2 + C) = \mathcal{L}^{\mathrm{M}}_\omega(\varphi_1, \varphi_2) + C$*.*

Proof. The proof is similar to that given in [2, 3]. □

3. Aubin-Yau functionals on compact complex manifolds

3.1. **The main idea.** The strategy to construct Aubin-Yau functionals is to use the inequalities (1.5) and (1.6) to determine the extra terms. Firstly we can show that

$$\begin{aligned} \frac{n}{n+1}\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) &= \frac{1}{V_\omega}\sum_{i=1}^{n-1}\frac{i}{n+1}\int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^i_\varphi \wedge \omega^{n-1-i} \\ &- \frac{\mathcal{A}_\omega(\varphi) + \mathcal{B}_\omega(\varphi)}{n+1} + \frac{\mathcal{C}_\omega(\varphi) + \mathcal{D}_\omega(\varphi)}{2}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} (n+1)\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) &= \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^i_\varphi \wedge \omega^{n-1-i} \\ &+ \frac{\mathcal{E}_\omega(\varphi) + \mathcal{F}_\omega(\varphi)}{2} + (n+1)\left[\mathcal{A}^1_\omega(\varphi) + \mathcal{B}^1_\omega(\varphi)\right] \\ &- \frac{\mathcal{A}^2_\omega(\varphi) + \mathcal{B}^2_\omega(\varphi)}{n-1}, \end{aligned} \tag{3.2}$$

where $\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi), \mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi), \mathcal{A}_\omega(\varphi), \mathcal{B}_\omega(\varphi), \mathcal{C}_\omega(\varphi), \mathcal{D}_\omega(\varphi), \mathcal{E}_\omega(\varphi), \mathcal{F}_\omega(\varphi), \mathcal{A}^1_\omega(\varphi), \mathcal{B}^1_\omega(\varphi),$ $\mathcal{A}^2_\omega(\varphi), \mathcal{B}^2_\omega(\varphi)$ are functionals determined in next subsection. Inspired by (3.1) and (3.2), we define Aubin-Yau functionals as follows:

$$\begin{aligned} \mathcal{I}^{\mathrm{AY}}_\omega(\varphi) &:= \mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) \\ &+ a^1_1\mathcal{A}^1_\omega(\varphi) + a^2_1\mathcal{A}^2_\omega(\varphi) + b^1_1\mathcal{B}^1_\omega(\varphi) + b^2_1\mathcal{B}^2_\omega(\varphi) \\ &+ c_1\mathcal{C}_\omega(\varphi) + d_1\mathcal{D}_\omega(\varphi) + e_1\mathcal{E}_\omega(\varphi) + f_1\mathcal{F}_\omega(\varphi), \end{aligned} \tag{3.3}$$

$$\begin{aligned} \mathcal{J}^{\mathrm{AY}}_\omega(\varphi) &= \mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) \\ &+ (a^1_2 - 1)\mathcal{A}^1_\omega(\varphi) + (a^2_2 - 1)\mathcal{A}^2_\omega(\varphi) + (b^1_2 - 1)\mathcal{B}^1_\omega(\varphi) + (b^2_2 - 1)\mathcal{B}^2_\omega(\varphi) \\ &+ c_2\mathcal{C}_\omega(\varphi) + d_2\mathcal{D}_\omega(\varphi) + e_2\mathcal{E}_\omega(\varphi) + f_2\mathcal{F}_\omega(\varphi). \end{aligned} \tag{3.4}$$

Here $a^i_j, b^i_j, c_k, d_k, e_k$, and f_k are constants determined by the following two inequalities:

$$\begin{aligned} \frac{n}{n+1}\mathcal{I}^{\mathrm{AY}}_\omega(\varphi) - \mathcal{J}^{\mathrm{AY}}_\omega(\varphi) &= \frac{1}{V_\omega}\sum_{i=1}^{n-1}\frac{i}{n+1}\int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^i_\varphi \wedge \omega^{n-1-i} \geq 0, \\ (n+1)\mathcal{J}^{\mathrm{AY}}_\omega(\varphi) - \mathcal{I}^{\mathrm{AY}}_\omega(\varphi) &= \sum_{i=1}^{n-1}\frac{n-1-i}{V_\omega}\int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^i_\varphi \wedge \omega^{n-1-i} \geq 0. \end{aligned}$$

It gives us a system of linear equations of 16 parameters and eventually we evaluate these parameters:

$$\begin{aligned}
a_1^1 &= b_1^1 = -\frac{n}{n-1}, \quad a_2^1 = b_2^1 = -\frac{n}{n^2-1},\\
a_1^2 &= b_1^2 = \frac{n}{(n-1)^2}, \quad a_2^2 = b_2^2 = \frac{n}{n+1}\left(1+\frac{n}{(n-1)^2}\right),\\
c_1 &= d_1 = -\frac{n+1}{2(n-1)}, \quad e_2 = f_2 = -\frac{n}{2(n^2-1)},\\
c_2 &= d_2 = e_1 = f_1 = -\frac{1}{2(n-1)}.
\end{aligned}$$

By a long computation we find an explicit and shorted formulae for $\mathcal{I}_\omega^{\mathrm{AY}}(\varphi)$ and $\mathcal{J}_\omega^{\mathrm{AY}}(\varphi)$:

$$\begin{aligned}
\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) &= \frac{1}{V_\omega}\int_X(\omega^n-\omega_\varphi^n)\\
&- \frac{n}{2V_\omega}\sum_{i=1}^{n-2}\int_X \varphi\omega_\varphi^i\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi - \frac{n}{2V_\omega}\int_X\varphi\omega^{n-2}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
&+ \frac{n}{2V_\omega}\sum_{i=1}^{n-2}\int_X \varphi\omega_\varphi^i\wedge\omega^{n-2-i}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi + \frac{n}{2V_\omega}\int_X\varphi\omega^{n-2}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi,
\end{aligned}\tag{3.5}$$

$$\begin{aligned}
\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) &= -\mathcal{L}_\omega^{\mathrm{M}}(\varphi)+\frac{1}{V_\omega}\int_X\varphi\omega^n\\
&- \frac{n}{2V_\omega}\sum_{i=1}^{n-2}\int_X \varphi\omega_\varphi^i\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi - \frac{n}{2V_\omega}\int_X\varphi\omega^{n-2}\wedge\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\\
&+ \frac{n}{2V_\omega}\sum_{i=1}^{n-2}\int_X \varphi\omega_\varphi^i\wedge\omega^{n-2-i}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi + \frac{n}{2V_\omega}\int_X\varphi\omega^{n-2}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi.
\end{aligned}\tag{3.6}$$

3.2. **The construction of Aubin-Yau functionals.** Let (X,g) be a compact complex manifold of the complex dimension $n\geq 3$ and ω be its associated real $(1,1)$-form. We recall some notation in [2]. For any $\varphi\in\mathcal{P}_\omega$ we set

$$\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) := \frac{1}{V_\omega}\int_X\varphi(\omega^n-\omega_\varphi^n),\tag{3.7}$$

$$\mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi) := \int_0^1\frac{\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(s\cdot\varphi)}{s}ds = \frac{1}{V_\omega}\int_0^1\int_X\varphi(\omega^n-\omega_{s\cdot\varphi}^n)ds.\tag{3.8}$$

Two relations showed in [2] are

$$\begin{aligned}
&\frac{n}{n+1}\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)-\mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)\\
&= \frac{1}{V_\omega}\int_X\varphi\cdot(-\sqrt{-1}\partial\overline{\partial}\varphi)\wedge\sum_{j=1}^{n-1}\frac{j}{n+1}\omega^{n-1-j}\wedge\omega_\varphi^j,
\end{aligned}\tag{3.9}$$

$$\begin{aligned}
&(n+1)\mathcal{J}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)-\mathcal{I}_{\omega|\bullet}^{\mathrm{AY}}(\varphi)\\
&= \frac{1}{V_\omega}\int_X\varphi\cdot(-\sqrt{-1}\partial\overline{\partial}\varphi)\wedge\sum_{j=0}^{n-1}(n-1-j)\omega^{n-1-j}\wedge\omega_\varphi^j.
\end{aligned}\tag{3.10}$$

According to the expression of $\mathcal{L}^{\mathrm{M}}_{\omega}(\varphi)$, we set

$$\mathcal{A}_{\omega}(\varphi) := \sum_{i=0}^{n-2}\frac{i+1}{2V_{\omega}}\int_X \varphi\omega_{\varphi}^{i}\wedge\omega^{n-2-i}\wedge -\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi, \tag{3.11}$$

$$\mathcal{B}_{\omega}(\varphi) := \sum_{i=0}^{n-2}\frac{i+1}{2V_{\omega}}\int_X \varphi\omega_{\varphi}^{i}\wedge\omega^{n-2-i}\wedge \sqrt{-1}\partial\omega\wedge\partial\varphi. \tag{3.12}$$

Using (3.9) we obtain

$$\begin{aligned}
&\frac{n}{n+1}\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)-\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)\\
=\ &\frac{1}{V_{\omega}}\int_X\sqrt{-1}\partial\left(\varphi\sum_{j=1}^{n-1}\frac{j}{n+1}\omega^{n-1-j}\wedge\omega_{\varphi}^{j}\right)\wedge\overline{\partial}\varphi\\
=\ &\frac{1}{V_{\omega}}\int_X\sqrt{-1}\left(\partial\varphi\wedge\sum_{j=1}^{n-1}\frac{j}{n+1}\omega^{n-1-j}\wedge\omega_{\varphi}^{j}\right)\wedge\overline{\partial}\varphi\\
+\ &\frac{1}{V_{\omega}}\int_X\sqrt{-1}\varphi\sum_{j=1}^{n-1}\frac{j}{n+1}\big[(n-1-j)\omega^{n-2-j}\wedge\partial\omega\wedge\omega_{\varphi}^{j}\\
+\ &\omega^{n-1-j}\wedge j\omega_{\varphi}^{j-1}\wedge\partial\omega\big]\wedge\overline{\partial}\varphi;
\end{aligned}$$

from the identity $i(n-1-i)+(i+1)^2=(i+1)+in$, it follows that

$$\begin{aligned}
&\frac{n}{n+1}\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)-\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)\\
=\ &\frac{1}{V_{\omega}}\sum_{i=1}^{n-1}\frac{i}{n+1}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{n-1-i}\wedge\omega_{\varphi}^{i}\\
+\ &\frac{1}{V_{\omega}}\sum_{i=1}^{n-2}\frac{i(n-1-i)}{n+1}\int_X\varphi\omega_{\varphi}^{i}\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
+\ &\frac{1}{V_{\omega}}\sum_{i=0}^{n-2}\frac{(i+1)^2}{n+1}\int_X\varphi\omega_{\varphi}^{i}\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
=\ &\frac{1}{V_{\omega}}\sum_{i=1}^{n-1}\frac{i}{n+1}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_{\varphi}^{i}\wedge\omega^{n-1-i}\\
+\ &\frac{1}{V_{\omega}}\sum_{i=1}^{n-2}\frac{i+1+in}{n+1}\int_X\varphi\omega_{\varphi}^{i}\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
+\ &\frac{1}{(n+1)V_{\omega}}\int_X\varphi\omega^{n-2}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi.
\end{aligned}$$

To simplify the notation, we set

$$\mathcal{C}_{\omega}(\varphi):=\frac{1}{V_{\omega}}\sum_{i=1}^{n-2}\frac{in}{n+1}\int_X\varphi\omega_{\varphi}^{i}\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi. \tag{3.13}$$

Since $n \geq 3$, the above expression is well defined. Therefore

$$\begin{aligned}
&\frac{n}{n+1}\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) \\
&= \frac{1}{V_\omega}\sum_{i=1}^{n-1}\frac{i}{n+1}\int_X \sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_\varphi\wedge\omega^{n-1-i} - \frac{2}{n+1}\mathcal{A}_\omega(\varphi) + \mathcal{C}_\omega(\varphi).
\end{aligned} \tag{3.14}$$

On the other hand, using the slightly different method, we obtain (see A.1)

$$\begin{aligned}
&\frac{n}{n+1}\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) \\
&= \frac{1}{V_\omega}\sum_{i=1}^{n-1}\frac{i}{n+1}\int_X \sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_\varphi\wedge\omega^{n-1-i} - \frac{2}{n+1}\mathcal{B}_\omega(\varphi) + \mathcal{D}_\omega(\varphi)
\end{aligned} \tag{3.15}$$

where

$$\mathcal{D}_\omega(\varphi) := \frac{1}{V_\omega}\sum_{i=1}^{n-2}\frac{in}{n+1}\int_X \varphi\omega^i_\varphi\wedge\omega^{n-2-i}\wedge -\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi. \tag{3.16}$$

Equations (3.14) and (3.15) implies

$$\begin{aligned}
\frac{n}{n+1}\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) &= \frac{1}{V_\omega}\sum_{i=1}^{n-1}\frac{i}{n+1}\int_X \sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_\varphi\wedge\omega^{n-1-i} \\
&- \frac{\mathcal{A}_\omega(\varphi)+\mathcal{B}_\omega(\varphi)}{n+1} + \frac{\mathcal{C}_\omega(\varphi)+\mathcal{D}_\omega(\varphi)}{2}.
\end{aligned} \tag{3.17}$$

By the definition we have

$$\begin{aligned}
\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) &= \frac{1}{V_\omega}\int_0^1\int_X(\varphi\omega^n - \varphi\omega^n_{s\varphi})ds = \frac{1}{V_\omega}\int_X\varphi\omega^n - \frac{1}{V_\omega}\int_0^1\int_X\varphi\omega^n_{t\varphi}dt \\
&= \frac{1}{V_\omega}\int_X\varphi\omega^n - (\mathcal{L}^{\mathrm{M}}_\omega(\varphi) - \mathcal{A}_\omega(\varphi) - \mathcal{B}_\omega(\varphi)) \\
&= \frac{1}{V_\omega}\int_X\varphi\omega^n - \mathcal{L}^{\mathrm{M}}_\omega(\varphi) + \mathcal{A}_\omega(\varphi) + \mathcal{B}_\omega(\varphi).
\end{aligned}$$

If we define

$$\mathcal{E}_\omega(\varphi) := \sum_{i=0}^{n-3}\frac{n^2}{V_\omega}\int_X\varphi\omega^i_\varphi\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi, \tag{3.18}$$

$$\mathcal{A}^1_\omega(\varphi) := \sum_{i=0}^{n-3}\frac{i+1}{2V_\omega}\int_X\varphi\omega^i_\varphi\wedge\omega^{n-2-i}\wedge -\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi \tag{3.19}$$

$$\mathcal{A}^2_\omega(\varphi) := \frac{n-1}{2V_\omega}\int_X\varphi\omega^{n-2}_\varphi\wedge -\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi, \tag{3.20}$$

then $\mathcal{A}^1_\omega(\varphi) + \mathcal{A}^2_\omega(\varphi) = \mathcal{A}_\omega(\varphi)$ and it follows that (see A.1)

$$\begin{aligned}
(n+1)\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet} - \mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) &= \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_\varphi\wedge\omega^{n-1-i} \\
&+ \mathcal{E}_\omega(\varphi) + 2(n+1)\mathcal{A}^1_\omega(\varphi) - \frac{2}{n-1}\mathcal{A}^2_\omega(\varphi).
\end{aligned} \tag{3.21}$$

Introduce the corresponding functionals

$$\mathcal{F}_\omega(\varphi) := \sum_{i=0}^{n-3}\frac{n^2}{V_\omega}\int_X \varphi\omega_\varphi^i\wedge\omega^{n-2-i}\wedge -\sqrt{-1}\partial\omega\wedge\partial\varphi, \tag{3.22}$$

$$\mathcal{B}^1_\omega(\varphi) := \sum_{i=0}^{n-3}\frac{i+1}{2V_\omega}\int_X \varphi\omega_\varphi^i\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\partial\varphi \tag{3.23}$$

$$\mathcal{B}^2_\omega(\varphi) := \frac{n-1}{2V_\omega}\int_X \varphi\omega_\varphi^{n-2}\wedge\sqrt{-1}\partial\omega\wedge\partial\varphi. \tag{3.24}$$

Then $\mathcal{B}^1_\omega(\varphi)+\mathcal{B}^2_\omega(\varphi)=\mathcal{B}_\omega(\varphi)$ and hence (see A.2)

$$\begin{aligned}(n+1)\mathcal{J}^{\rm AY}_{\omega|\bullet}(\varphi)-\mathcal{I}^{\rm AY}_{\omega|\bullet}(\varphi) &= \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\varphi^i\wedge\omega^{n-1-i}\\ &+ \mathcal{F}_\omega(\varphi)+2(n+1)\mathcal{B}^1_\omega(\varphi)-\frac{2}{n-1}\mathcal{B}^2_\omega(\varphi).\end{aligned} \tag{3.25}$$

The equations (3.21) and (3.25) together gives

$$\begin{aligned}(n+1)\mathcal{J}^{\rm AY}_{\omega|\bullet}(\varphi)-\mathcal{I}^{\rm AY}_{\omega|\bullet}(\varphi) &= \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\varphi^i\wedge\omega^{n-1-i}\\ &+ \frac{\mathcal{E}_\omega(\varphi)+\mathcal{F}_\omega(\varphi)}{2}+(n+1)(\mathcal{A}^1_\omega(\varphi)+\mathcal{B}^1_\omega(\varphi)\\ &- \frac{\mathcal{A}^2_\omega(\varphi)+\mathcal{B}^2_\omega(\varphi)}{n-1}.\end{aligned} \tag{3.26}$$

Now, we define Aubin-Yau functionals over any compact complex manifolds as follows:

$$\begin{aligned}\mathcal{I}^{\rm AY}_\omega(\varphi) &:= \mathcal{I}^{\rm AY}_{\omega|\bullet}(\varphi)\\ &+ a^1_1\mathcal{A}^1_\omega(\varphi)+a^2_1\mathcal{A}^2_\omega(\varphi)+b^1_1\mathcal{B}^1_\omega(\varphi)+b^2_1\mathcal{B}^2_\omega(\varphi)\\ &+ c_1\mathcal{C}_\omega(\varphi)+d_1\mathcal{D}_\omega(\varphi)+e_1\mathcal{E}_\omega(\varphi)+f_1\mathcal{F}_\omega(\varphi),\end{aligned} \tag{3.27}$$

$$\begin{aligned}\mathcal{J}^{\rm AY}_\omega(\varphi) &:= -\mathcal{L}^{\rm M}_\omega(\varphi)+\frac{1}{V_\omega}\int_X\varphi\omega^n\\ &+ a^1_2\mathcal{A}^1_\omega(\varphi)+a^2_2\mathcal{A}^2_\omega(\varphi)+b^1_2\mathcal{B}^1_\omega(\varphi)+b^2_2\mathcal{B}^2_\omega(\varphi)\\ &+ c_2\mathcal{C}_\omega(\varphi)+d_2\mathcal{D}_\omega(\varphi)+e_2\mathcal{E}_\omega(\varphi)+f_2\mathcal{F}_\omega(\varphi),\\ &= \mathcal{J}^{\rm AY}_{\omega|\bullet}(\varphi)\\ &+ (a^1_2-1)\mathcal{A}^1_\omega(\varphi)+(a^2_2-1)\mathcal{A}^2_\omega(\varphi)+(b^1_2-1)\mathcal{B}^1_\omega(\varphi)+(b^2_2-1)\mathcal{B}^2_\omega(\varphi)\\ &+ c_2\mathcal{C}_\omega(\varphi)+d_2\mathcal{D}_\omega(\varphi)+e_2\mathcal{E}_\omega(\varphi)+f_2\mathcal{F}_\omega(\varphi).\end{aligned} \tag{3.28}$$

Plugging (3.27) and (3.28) into (3.26) and (3.17), we obtain

$$\frac{n}{n+1}\mathcal{I}^{\rm AY}_\omega(\varphi)-\mathcal{J}^{\rm AY}_\omega(\varphi)=\frac{1}{V_\omega}\sum_{i=1}^{n-1}\frac{i}{n+1}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\varphi^i\wedge\omega^{n-1-i}\geq 0, \tag{3.29}$$

and

$$(n+1)\mathcal{J}^{\rm AY}_\omega(\varphi)-\mathcal{I}^{\rm AY}_\omega(\varphi)=\sum_{i=0}^{n-1}\frac{n-1-i}{V_\omega}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\varphi^i\wedge\omega^{n-1-i}\geq 0, \tag{3.30}$$

where we require that constants satisfy the following linear equations system

$$\frac{n}{n+1}a_1^1 - (a_2^1 - 1) = \frac{1}{n+1}, \quad \frac{n}{n+1}a_1^2 - (a_2^2 - 1) = \frac{1}{n+1}, \tag{3.31}$$

$$\frac{n}{n+1}b_1^1 - (b_2^1 - 1) = \frac{1}{n+1}, \quad \frac{n}{n+1}b_1^2 - (b_2^2 - 1) = \frac{1}{n+1}, \tag{3.32}$$

$$\frac{n}{n+1}c_1 - c_2 = -\frac{1}{2}, \quad \frac{n}{n+1}d_1 - d_2 = -\frac{1}{2}, \tag{3.33}$$

$$\frac{n}{n+1}e_1 - e_2 = 0, \quad \frac{n}{n+1}f_1 - f_2 = 0, \tag{3.34}$$

$$(n+1)(a_2^1 - 1) - a_1^1 = -(n+1), \quad (n+1)(a_2^2 - 1) - a_1^2 = \frac{1}{n-1}, \tag{3.35}$$

$$(n+1)(b_2^1 - 1) - b_1^1 = -(n+1), \quad (n+1)(b_2^2 - 1) - b_1^2 = \frac{1}{n-1}, \tag{3.36}$$

$$(n+1)c_2 - c_1 = 0, \quad (n+1)d_2 - d_1 = 0, \tag{3.37}$$

$$(n+1)e_2 - e_1 = -\frac{1}{2}, \quad (n+1)f_2 - f_1 = -\frac{1}{2}. \tag{3.38}$$

The constants $a_i^j, b_i^j, c_i, d_i, e_i$ and f_i , calculated in Appendix B, are

$$a_1^1 = b_1^1 = -\frac{n}{n-1}, \quad a_2^1 = b_2^1 = -\frac{n}{n^2-1}, \tag{3.39}$$

$$a_1^2 = b_1^2 = \frac{n}{(n-1)^2}, \quad a_2^2 = b_2^2 = \frac{n}{n+1}\left(1 + \frac{n}{(n-1)^2}\right) \tag{3.40}$$

$$c_1 = d_1 = -\frac{n+1}{2(n-1)}, \quad e_2 = f_2 = -\frac{n}{2(n^2-1)}, \tag{3.41}$$

$$c_2 = d_2 = e_1 = f_1 = -\frac{1}{2(n-1)}. \tag{3.42}$$

The explicit formulas for $\mathcal{I}_\omega^{\mathrm{AY}}(\varphi)$ and $\mathcal{J}_\omega^{\mathrm{AY}}(\varphi)$ are given in Proposition C.1 and C.2 respectively. Namely,

$$\begin{aligned}
\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) &= \frac{1}{V_\omega}\int_X \varphi(\omega^n - \omega_\varphi^n) - \frac{n}{2V_\omega}\sum_{i=0}^{n-2}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&+ \frac{n}{2V_\omega}\sum_{i=0}^{n-2}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi, \\
&= \frac{1}{V_\omega}\sum_{i=0}^{n-1}\int_X \sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-1-i},
\end{aligned} \tag{3.43}$$

$$\begin{aligned}
\mathcal{J}_\omega^{\mathrm{AY}}(\varphi) &= -\mathcal{L}_\omega^{\mathrm{M}}(\varphi) + \frac{1}{V_\omega}\int_X \varphi\omega^n - \frac{n}{2V_\omega}\sum_{i=0}^{n-2}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&+ \frac{n}{2V_\omega}\sum_{i=0}^{n-2}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&= \frac{1}{V_\omega}\sum_{i=0}^{n-1}\int_X \frac{n-i}{n+1}\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^i \wedge \omega^{n-i-1}.
\end{aligned} \tag{3.44}$$

Here the formulas (3.43) and (3.44) come from the solution of the system of linear equations (3.29) and (3.30).

From (3.29), (3.30), (3.37) and (3.44), we deduce the following

Theorem 3.1. *For any $\varphi \in \mathcal{P}_\omega$, one has*

$$\frac{n}{n+1}\mathcal{I}^{\mathrm{AY}}_\omega(\varphi) - \mathcal{J}^{\mathrm{AY}}_\omega(\varphi) \geq 0, \tag{3.45}$$

$$(n+1)\mathcal{J}^{\mathrm{AY}}_\omega(\varphi) - \mathcal{I}^{\mathrm{AY}}_\omega(\varphi) \geq 0. \tag{3.46}$$

In particular

$$\frac{1}{n+1}\mathcal{I}^{\mathrm{AY}}_\omega(\varphi) \leq \mathcal{J}^{\mathrm{AY}}_\omega(\varphi) \leq \frac{n}{n+1}\mathcal{I}^{\mathrm{AY}}_\omega(\varphi), \tag{3.47}$$

$$\frac{n+1}{n}\mathcal{J}^{\mathrm{AY}}_\omega(\varphi) \leq \mathcal{I}^{\mathrm{AY}}_\omega(\varphi) \leq (n+1)\mathcal{J}^{\mathrm{AY}}_\omega(\varphi), \tag{3.48}$$

$$\frac{1}{n}\mathcal{J}^{\mathrm{AY}}_\omega(\varphi) \leq \frac{1}{n+1}\mathcal{J}^{\mathrm{AY}}_\omega(\varphi) \leq \mathcal{I}^{\mathrm{AY}}_\omega(\varphi) - \mathcal{J}^{\mathrm{AY}}_\omega(\varphi) \tag{3.49}$$

$$\leq \frac{n}{n+1}\mathcal{I}^{\mathrm{AY}}_\omega(\varphi) \leq n\mathcal{J}^{\mathrm{AY}}_\omega(\varphi). \tag{3.50}$$

Appendix A. Proof the identities (3.14), (3.21) and (3.25)

In Appendix A we verify the identities (3.14), (3.21) and (3.25).

$$\begin{aligned}
&\frac{n}{n+1}\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) - \mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) \\
= \ & \frac{1}{V_\omega}\int_X \left(\varphi\cdot\sum_{j=1}^{n-1}\frac{j}{n+1}\omega^{n-1-j}\wedge\omega^j_\varphi\right)\wedge\sqrt{-1}\overline{\partial}\partial\varphi \\
= \ & \frac{1}{V_\omega}\int_X -\sqrt{-1}\overline{\partial}\left(\varphi\sum_{j=1}^{n-1}\frac{j}{n+1}\omega^{n-1-j}\wedge\omega^j_\varphi\right)\wedge\partial\varphi \\
= \ & \frac{1}{V_\omega}\int_X -\sqrt{-1}\left(\overline{\partial}\varphi\wedge\sum_{j=1}^{n-1}\frac{j}{n+1}\omega^{n-1-j}\wedge\omega^j_\varphi\right)\wedge\partial\varphi \\
+ \ & \frac{1}{V_\omega}\int_X -\sqrt{-1}\varphi\sum_{j=1}^{n-1}\frac{j}{n+1}\left[(n-1-j)\omega^{n-2-j}\wedge\overline{\partial}\omega\wedge\omega^j_\varphi\right. \\
+ \ & \left.\omega^{n-1-j}\wedge j\omega^{j-1}_\varphi\wedge\overline{\partial}\omega\right]\wedge\partial\varphi \\
= \ & \frac{1}{V_\omega}\sum_{i=1}^{n-1}\frac{i}{n+1}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{n-1-i}\wedge\omega^i_\varphi \\
+ \ & \frac{1}{V_\omega}\sum_{i=1}^{n-2}\frac{i(n-1-i)}{n+1}\int_X -\sqrt{-1}\varphi\omega^{n-2-i}\wedge\omega^i_\varphi\wedge\overline{\partial}\omega\wedge\partial\varphi \\
+ \ & \frac{1}{V_\omega}\sum_{i=0}^{n-2}\frac{(i+1)^2}{n+1}\int_X -\sqrt{-1}\varphi\omega^{n-2-i}\wedge\omega^i_\varphi\wedge\overline{\partial}\omega\wedge\partial\varphi \\
= \ & \frac{1}{V_\omega}\sum_{i=1}^{n-1}\frac{i}{n+1}\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{n-1-i}\wedge\omega^i_\varphi - \frac{2}{n+1}\mathcal{B}_\omega(\varphi) + \mathcal{D}_\omega(\varphi)
\end{aligned} \tag{A.1}$$

which gives (3.15). Calculate

$$
\begin{aligned}
& (n+1)\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet} - \mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) \\
= \quad & \frac{1}{V_\omega}\int_X \left(\varphi\sum_{j=0}^{n-1}(n-1-j)\omega^{n-1-j}\wedge\omega^j_\varphi\right)\wedge(-\sqrt{-1}\partial\overline{\partial}\varphi) \\
= \quad & \frac{1}{V_\omega}\int_X \sqrt{-1}\partial\left(\varphi\sum_{j=0}^{n-1}(n-1-j)\omega^{n-1-j}\wedge\omega^j_\varphi\right)\wedge\overline{\partial}\varphi \\
= \quad & \frac{1}{V_\omega}\int_X \sqrt{-1}\partial\varphi\wedge\sum_{j=0}^{n-1}(n-1-j)\omega^{n-1-j}\wedge\omega^j_\varphi\wedge\overline{\partial}\varphi \\
+ \quad & \frac{1}{V_\omega}\int_X \sqrt{-1}\varphi\sum_{j=0}^{n-1}\big[(n-1-j)^2\omega^{n-2-j}\wedge\partial\omega\wedge\omega^j_\varphi \\
+ \quad & (n-1-j)j\omega^{n-1-j}\wedge\omega^{j-1}_\varphi\wedge\partial\omega\big]\wedge\overline{\partial}\varphi \\
= \quad & \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-j)\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_\varphi\wedge\omega^{n-1-i} \\
+ \quad & \frac{1}{V_\omega}\sum_{j=0}^{n-1}(n-1-j)^2\int_X\varphi\omega^{n-2-j}\wedge\omega^j_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi \\
+ \quad & \frac{1}{V_\omega}\sum_{j=0}^{n-1}(n-1-j)j\int_X\varphi\omega^{n-1-j}\wedge\omega^{j-1}_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi \\
= \quad & \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-j)\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_\varphi\wedge\omega^{n-1-i} \\
+ \quad & \frac{1}{V_\omega}\sum_{j=0}^{n-2}(n-1-j)^2\int_X\varphi\omega^{n-2-j}\wedge\omega^j_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi \\
+ \quad & \frac{1}{V_\omega}\sum_{i=0}^{n-3}(i+1)(n-i-2)\int_X\varphi\omega^{n-2-i}\wedge\omega^i_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi \\
= \quad & \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_\varphi\wedge\omega^{n-1-i} \\
+ \quad & \frac{1}{V_\omega}\sum_{i=0}^{n-3}[n^2-(n+1)(i+1)]\int_X\varphi\omega^{n-2-i}\wedge\omega^i_\varphi\wedge(\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi) \\
+ \quad & \frac{1}{V_\omega}\int_X\varphi\omega^{n-2}_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi
\end{aligned}
$$

where we use the elementary identity

$$
\begin{aligned}
& (n-1-i)^2+(i+1)(n-i-2) \\
= \quad & (n-1)^2+i^2-2(n-1)i+(n-2)(i+1)-i(i+1) \\
= \quad & n^2-2n+1+i^2-2ni+2i+ni+n-2i-2-i^2-i \\
= \quad & n^2-n-1-ni-i \quad = \quad -(n+1)(i+1)+n^2.
\end{aligned}
$$

Using the definitions of $\mathcal{E}_\omega(\varphi), \mathcal{A}^1_\omega(\varphi)$, $\mathcal{A}^2_\omega(\varphi)$, we have $\mathcal{A}^1_\omega(\varphi)+\mathcal{A}^2_\omega(\varphi)=\mathcal{A}_\omega(\varphi)$ and hence (3.21) holds. Similarly, we have $\mathcal{B}^1_\omega(\varphi)+\mathcal{B}^2_\omega(\varphi)=\mathcal{B}_\omega(\varphi)$ and

$$
\begin{aligned}
& (n+1)\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)-\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) \\
= \;& \frac{1}{V_\omega}\int_X -\sqrt{-1}\partial\left(\varphi\sum_{j=0}^{n-1}(n-1-j)\omega^{n-1-j}\wedge\omega^j_\varphi\right)\wedge\partial\varphi \\
= \;& \frac{1}{V_\omega}\int_X -\sqrt{-1}\partial\varphi\wedge\sum_{j=0}^{n-1}(n-1-j)\omega^{n-1-j}\wedge\omega^j_\varphi\wedge\partial\varphi \\
+ \;& \frac{1}{V_\omega}\int_X -\sqrt{-1}\varphi\sum_{j=0}^{n-1}(n-1-j)((n-1-j)\omega^{n-2-j}\wedge\overline{\partial}\omega\wedge\omega^j_\varphi \\
+ \;& j\omega^{n-1-j}\wedge\omega^{j-1}_\varphi\wedge\overline{\partial}\omega)\wedge\partial\varphi \\
= \;& \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{n-1-i}\wedge\omega^i_\varphi \\
+ \;& \frac{1}{V_\omega}\sum_{j=0}^{n-1}(n-1-j)^2\int_X\varphi\omega^{n-2-j}\wedge\omega^j_\varphi\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi) \\
+ \;& \frac{1}{V_\omega}\sum_{j=0}^{n-1}(n-1-j)j\int_X\varphi\omega^{n-1-j}\wedge\omega^{j-1}_\varphi\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi) \\
= \;& \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{n-1-i}\wedge\omega^i_\varphi \\
+ \;& \frac{1}{V_\omega}\sum_{i=0}^{n-2}(n-1-i)^2\int_X\varphi\omega^{n-2-i}\wedge\omega^i_\varphi\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi) \\
+ \;& \frac{1}{V_\omega}\sum_{i=0}^{n-3}(n-i-2)(i+1)\int_X\varphi\omega^{n-2-i}\wedge\omega^i_\varphi\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi) \\
= \;& \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{n-1-i}\wedge\omega^i_\varphi \\
+ \;& \frac{1}{V_\omega}\sum_{i=0}^{n-3}[n^2-(n+1)(i+1)]\int_X\varphi\omega^{n-2-i}\wedge\omega^i_\varphi\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi) \\
+ \;& \frac{1}{V_\omega}\int_X\varphi\omega^{n-2}_\varphi\wedge(-\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi).
\end{aligned}
$$

and hence

$$
\begin{aligned}
(n+1)\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)-\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) \;=\;& \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_\varphi\wedge\omega^{n-1-i} \\
+\;& \mathcal{F}_\omega(\varphi)+2(n+1)\mathcal{B}^1_\omega(\varphi)-\frac{2}{n-1}\mathcal{B}^2_\omega(\varphi). \qquad \text{(A.2)}
\end{aligned}
$$

Therefore (3.21) and (3.25) together gives

$$
\begin{aligned}
(n+1)\mathcal{J}^{\mathrm{AY}}_{\omega|\bullet}(\varphi)-\mathcal{I}^{\mathrm{AY}}_{\omega|\bullet}(\varphi) &= \frac{1}{V_\omega}\sum_{i=0}^{n-1}(n-1-i)\int_X\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i_\varphi\wedge\omega^{n-1-i}\\
&+ \frac{\mathcal{E}_\omega(\varphi)+\mathcal{F}_\omega(\varphi)}{2}+(n+1)(\mathcal{A}^1_\omega(\varphi)+\mathcal{B}^1_\omega(\varphi)\\
&- \frac{\mathcal{A}^2_\omega(\varphi)+\mathcal{B}^2_\omega(\varphi)}{n-1}.
\end{aligned}\tag{A.3}
$$

Appendix B. Solve the system of the linear equations

In this section we try to solve the system of the linear equations (3.31)-(3.38). Firstly we solve (3.31) and (3.35) as follows: (3.31) and (3.35) gives us the following equations

$$
\begin{aligned}
\frac{n}{n+1}a^1_1-\frac{1}{n+1} &= a^1_2-1, \qquad\qquad \text{(B.1)}\\
(n+1)(a^1_2-1)+(n+1) &= a^1_1,\\
\frac{n}{n+1}a^2_1-\frac{1}{n+1} &= a^2_2-1, \qquad\qquad \text{(B.2)}\\
(n+1)(a^2_2-1)-\frac{1}{n-1} &= a^2_1.
\end{aligned}
$$

Plugging the first equation into second equation in (B.1), we have

$$
(n+1)\left(\frac{n}{n+1}a^1_1-\frac{1}{n+1}\right)+(n+1)=a^1_1
$$

which implies

$$
a^1_1=-\frac{n}{n-1},\quad a^1_2=-\frac{n}{n^2-1}.\tag{B.3}
$$

Similarly,

$$
(n+1)\left(\frac{n}{n+1}a^2_1-\frac{1}{n+1}\right)-\frac{1}{n+1}=a^2_1,
$$

therefore

$$
a^2_1=\frac{n}{(n-1)^2},\quad a^2_2=\frac{n}{n+1}\left(1+\frac{n}{(n-1)^2}\right)=\frac{n^3-n^2+n}{n^3-n^2-n+1}.\tag{B.4}
$$

Secondly, (3.32) and (3.36) implies

$$
\begin{aligned}
\frac{n}{n+1}b^1_1-\frac{1}{n+1} &= b^1_2-1, \qquad\qquad \text{(B.5)}\\
(n+1)(b^1_2-1) &= {}^1_1-(n+1),\\
\frac{n}{n+1}b^2_1-\frac{1}{n+1} &= b^2_2-1, \qquad\qquad \text{(B.6)}\\
(n+1)(b^2_2-1) &= b^2_1+\frac{1}{n-1}.
\end{aligned}
$$

The above linear equations system gives

$$
(n+1)\left(\frac{n}{n+1}b^1_1-\frac{1}{n+1}\right)=b^1_1-(n+1)
$$

and

$$(n+1)\left(\frac{n}{n+1}b_1^2 - \frac{1}{n+1}\right) = b_1^2 + \frac{1}{n-1},$$

respectively. Hence

$$\begin{aligned}
b_1^1 &= -\frac{n}{n-1}, && \text{(B.7)}\\
b_2^1 &= -\frac{n}{n^2-1},\\
b_1^2 &= \frac{n}{(n-1)^2}, && \text{(B.8)}\\
b_2^2 &= \frac{n}{n+1}\left(1+\frac{n}{(n-1)^2}\right).
\end{aligned}$$

Continuously, equations (3.33) and (3.37) shows that

$$\begin{aligned}
\frac{n}{n+1}c_1 - c_2 &= -\frac{1}{2}, \quad (n+1)c_2 - c_1 = 0,\\
\frac{n}{n+1}d_1 - d_2 &= -\frac{1}{2}, \quad (n+1)d_2 - d_1 = 0.
\end{aligned}$$

Eliminating c_2 and d_2 respectively, we have

$$\begin{aligned}
(n+1)\left(\frac{n}{n+1}c_1 + \frac{1}{2}\right) - c_1 &= 0,\\
(n+1)\left(\frac{n}{n+1}d_1 + \frac{1}{2}\right) - d_1 &= 0.
\end{aligned}$$

Thus

$$\begin{aligned}
c_1 &= -\frac{n+1}{2(n-1)}, && \text{(B.9)}\\
c_2 &= -\frac{1}{2(n-1)},\\
d_1 &= -\frac{n+1}{2(n-1)}, && \text{(B.10)}\\
d_2 &= -\frac{1}{2(n-1)}.
\end{aligned}$$

Similarly, from (3.34) and (3.38) we obtain

$$\begin{aligned}
\frac{n}{n+1}e_1 - e_2 = 0, \quad (n+1)e_2 - e_1 &= -\frac{1}{2},\\
\frac{n}{n+1}f_1 - f_2 = 0, \quad (n+1)f_2 - f_1 &= -\frac{1}{2},
\end{aligned}$$

and hence

$$e_1 = f_1 = -\frac{1}{2(n-1)}, \tag{B.11}$$

$$e_2 = f_2 = -\frac{n}{2(n^2-1)}. \tag{B.12}$$

Appendix C. Explicit formulas of $\mathcal{I}_\omega^{\mathrm{AY}}(\varphi)$ and $\mathcal{J}_\omega^{\mathrm{AY}}(\varphi)$

In this section we give the explicit formulas of $\mathcal{I}_\omega^{\mathrm{AY}}(\varphi)$ and $\mathcal{J}_\omega^{\mathrm{AY}}(\varphi)$. In what follows, we assume that $n \geq 3$. Using the constants determined in Appendix B, we have

$$
\begin{aligned}
\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) &= \frac{1}{V_\omega}\int_X \varphi(\omega^n - \omega_\varphi^n) \\
&= \frac{n}{n-1}\sum_{i=0}^{n-3}\frac{i+1}{2V_\omega}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&- \frac{n}{n-1}\frac{1}{2V_\omega}\int_X \varphi\omega_\varphi^{n-2} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&- \frac{n}{n-1}\sum_{i=0}^{n-3}\frac{i+1}{2V_\omega}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&+ \frac{n}{n-1}\frac{1}{2V_\omega}\int_X \varphi\omega_\varphi^{n-2} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&- \frac{n}{n-1}\sum_{i=1}^{n-2}\frac{i}{2V_\omega}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-3-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&+ \frac{n}{n-1}\sum_{i=1}^{n-2}\frac{i}{2V_\omega}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&- \frac{n^2}{n-1}\sum_{i=0}^{n-3}\frac{1}{2V_\omega}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&+ \frac{n^2}{n-1}\sum_{i=0}^{n-3}\frac{1}{2V_\omega}\int_X \varphi\omega_\varphi^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi.
\end{aligned}
$$

When $n = 3$, it is easy to see that

$$
\begin{aligned}
\mathcal{I}_\omega^{\mathrm{AY}}(\varphi) &= \frac{1}{V_\omega}\int_X \varphi(\omega^3 - \omega_\varphi^3) \\
&+ \frac{3}{4V_\omega}\int_X \varphi\omega \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi - \frac{3}{4V_\omega}\int_X \varphi\omega_\varphi \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
&- \frac{3}{4V_\omega}\int_X \varphi\omega \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi + \frac{3}{4V_\omega}\int_X \varphi\omega_\varphi \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&- \frac{3}{4V_\omega}\int_X \varphi\omega_\varphi \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{3}{4V_\omega}\int_X \varphi\omega_\varphi \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&- \frac{9}{4V_\omega}\int_X \varphi\omega \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{9}{4V_\omega}\int_X \varphi\omega \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&= \frac{1}{V_\omega}\int_X \varphi(\omega^3 - \omega_\varphi^3) \\
&- \frac{3}{2V_\omega}\int_X \varphi\omega \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{3}{2V_\omega}\int_X \varphi\omega \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
&- \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{3}{2V_\omega}\int_X \varphi\omega_\varphi \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi.
\end{aligned}
$$

For general $n \geq 4$, a simple computation shows

$$
\begin{aligned}
\mathcal{I}^{\mathrm{AY}}_{\omega}(\varphi) &= \frac{1}{V_\omega}\int_X \varphi(\omega^n - \omega^n_\varphi)\\
&+ \sum_{i=1}^{n-3}\frac{1}{2V_\omega}\left[\frac{n(i+1)}{n-1} - \frac{in}{n-1} - \frac{n^2}{n-1}\right]\int_X \varphi\omega^i_\varphi\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
&+ \frac{n}{n-1}\frac{1}{2V_\omega}\int_X \varphi\omega^{n-2}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
&- \frac{n(n-2)}{n-1}\frac{1}{2V_\omega}\int_X \varphi\omega^{n-2}_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
&- \frac{n}{n-1}\frac{1}{2V_\omega}\int_X \varphi\omega^{n-2}_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
&+ \sum_{i=1}^{n-3}\frac{1}{2V_\omega}\left[-\frac{n(i+1)}{n-1} + \frac{in}{n-1} + \frac{n^2}{n-1}\right]\int_X \varphi\omega^i_\varphi\wedge\omega^{n-2-i}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\\
&- \frac{n}{n-1}\frac{1}{2V_\omega}\int_X \varphi\omega^{n-2}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\\
&+ \frac{n(n-2)}{n-1}\frac{2}{2V_\omega}\int_X \varphi\omega^{n-2}_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\\
&+ \frac{n}{n-1}\frac{1}{2V_\omega}\int_X \varphi\omega^{n-2}_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\\
&- \frac{n^2}{n-1}\frac{1}{2V_\omega}\int_X \varphi\omega^{n-2}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \frac{n^2}{n-1}\frac{1}{2V_\omega}\int_X \varphi\omega^{n-2}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\\
&= \frac{1}{V_\omega}\int_X \varphi(\omega^n - \omega^n_\varphi)\\
&- \sum_{i=1}^{n-3}\frac{n}{2V_\omega}\int_X \varphi\omega^i_\varphi\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
&+ \sum_{i=1}^{n-3}\frac{n}{2V_\omega}\int_X \varphi\omega^i_\varphi\wedge\omega^{n-2-i}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\\
&- \frac{n}{2V_\omega}\int_X \varphi\omega^{n-2}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \frac{n}{2V_\omega}\int_X \varphi\omega^{n-2}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\\
&- \frac{n}{2V_\omega}\int_X \varphi\omega^{n-2}_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi + \frac{n}{2V_\omega}\int_X \varphi\omega^{n-2}_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi.
\end{aligned}
$$

Thus

Proposition C.1. *If $n \geq 3$, one has*

$$
\begin{aligned}
\mathcal{I}^{\mathrm{AY}}_{\omega}(\varphi) &= \frac{1}{V_\omega}\int_X \varphi(\omega^n - \omega^n_\varphi)\\
&- \frac{n}{2V_\omega}\sum_{i=1}^{n-2}\int_X \varphi\omega^i_\varphi\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi - \frac{n}{2V_\omega}\int_X \varphi\omega^{n-2}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
&+ \frac{n}{2V_\omega}\sum_{i=1}^{n-2}\int_X \varphi\omega^i_\varphi\wedge\omega^{n-2-i}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi + \frac{n}{2V_\omega}\int_X \varphi\omega^{n-2}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& \mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi) \\
= \quad & -\mathcal{L}^{\mathrm{M}}_{\omega}(\varphi) + \frac{1}{V_{\omega}}\int_X \varphi\omega^n \\
+ \quad & \frac{n}{n^2-1}\sum_{i=0}^{n-3}\frac{i+1}{2V_{\omega}}\int_X \varphi\omega_{\varphi}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
- \quad & \frac{n}{n+1}\left(n-1+\frac{n}{n-1}\right)\frac{1}{2V_{\omega}}\int_X \varphi\omega_{\varphi}^{n-2} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
- \quad & \frac{n}{n^2-1}\sum_{i=0}^{n-3}\frac{i+1}{2V_{\omega}}\int_X \varphi\omega_{\varphi}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
+ \quad & \frac{n}{n+1}\left(n-1+\frac{n}{n-1}\right)\frac{2}{2V_{\omega}}\int_X \varphi\omega_{\varphi}^{n-2} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
- \quad & \frac{n}{n^2-1}\sum_{i=1}^{n-2}\frac{i}{2V_{\omega}}\int_X \varphi\omega_{\varphi}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
+ \quad & \frac{n}{n^2-1}\sum_{i=1}^{n-2}\frac{i}{2V_{\omega}}\int_X \varphi\omega_{\varphi}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
- \quad & \frac{n^3}{n^2-1}\sum_{i=0}^{n-3}\frac{1}{2V_{\omega}}\int_X \varphi\omega_{\varphi}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
+ \quad & \frac{n^3}{n^2-1}\sum_{i=0}^{n-3}\frac{1}{2V_{\omega}}\int_X \varphi\omega_{\varphi}^i \wedge \omega^{n-2-i} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi
\end{aligned}
$$

When $n=3$, we have

$$
\begin{aligned}
& \mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi) \\
= \quad & -\mathcal{L}^{\mathrm{M}}_{\omega}(\varphi) + \frac{1}{V_{\omega}}\int_X \varphi\omega^3 \\
+ \quad & \frac{3}{8}\frac{1}{2V_{\omega}}\int_X \varphi\omega \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi - \frac{3}{4}\left(2+\frac{3}{2}\right)\frac{1}{2V_{\omega}}\int_X \varphi\omega_{\varphi} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi \\
- \quad & \frac{3}{8}\frac{1}{2V_{\omega}}\int_X \varphi\omega \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi + \frac{3}{4}\left(2+\frac{3}{2}\right)\frac{1}{2V_{\omega}}\int_X \varphi\omega_{\varphi} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
- \quad & \frac{3}{8}\frac{1}{2V_{\omega}}\int_X \varphi\omega_{\varphi} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{3}{8}\frac{1}{2V_{\omega}}\int_X \varphi\omega_{\varphi} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
- \quad & \frac{27}{8}\frac{1}{2V_{\omega}}\int_X \varphi\omega \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{27}{8}\frac{1}{2V_{\omega}}\int_X \varphi\omega \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
= \quad & -\mathcal{L}^{\mathrm{M}}_{\omega}(\varphi) + \frac{1}{V_{\omega}}\int_X \varphi\omega^3 \\
- \quad & \frac{3}{2V_{\omega}}\int_X \varphi\omega \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{3}{2V_{\omega}}\int_X \varphi\omega \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi \\
- \quad & \frac{3}{2V_{\omega}}\int_X \varphi\omega_{\varphi} \wedge \sqrt{-1}\partial\omega \wedge \overline{\partial}\varphi + \frac{3}{2V_{\omega}}\int_X \varphi\omega_{\varphi} \wedge \sqrt{-1}\overline{\partial}\omega \wedge \partial\varphi
\end{aligned}
$$

When $n\geq 4$, we have

$$\begin{aligned}
\mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi) &= -\mathcal{L}^{\mathrm{M}}_{\omega}(\varphi)+\frac{1}{V_\omega}\int_X\varphi\omega^n\\
&+ \frac{1}{2V_\omega}\sum_{i=1}^{n-3}\left[\frac{n(i+1)}{n^2-1}-\frac{in}{n^2-1}-\frac{n^3}{n^2-1}\right]\int_X\varphi\omega^i_\varphi\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
&+ \frac{n}{n^2-1}\frac{1}{2V_\omega}\int_X\varphi\omega^{n-2}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
&- \frac{n}{n+1}\left(n-1+\frac{n}{n-1}\right)\frac{1}{2V_\omega}\int_Z\varphi\omega^{n-2}_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
&+ \frac{1}{2V_\omega}\sum_{i=1}^{n-3}\left[-\frac{n(i+1)}{n^2-1}+\frac{in}{n^2-1}+\frac{n^3}{n^2-1}\right]\int_X\varphi\omega^i_\varphi\wedge\omega^{n-2-i}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\\
&- \frac{n}{n^2-1}\frac{1}{2V_\omega}\int_X\varphi\omega^{n-2}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\\
&+ \frac{n}{n+1}\left(n-1+\frac{n}{n-1}\right)\frac{1}{2V_\omega}\int_X\varphi\omega^{n-2}_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\\
&- \frac{n(n-2)}{(n^2-1)2V_\omega}\int_Z\varphi\omega^{n-2}_\varphi\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi+\frac{n(n-2)}{(n^2-1)2V_\omega}\int_X\varphi\omega^{n-2}_\varphi\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi\\
&- \frac{n^3}{n^2-1}\frac{1}{2V_\omega}\int_X\varphi\omega^{n-2}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi+\frac{n^3}{n^2-1}\frac{1}{2V_\omega}\int_X\varphi\omega^{n-2}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi
\end{aligned}$$

Using the identities

$$\begin{aligned}
\frac{n(i+1)}{n^2-1}-\frac{in}{n^2-1}-\frac{n^3}{n^2-1} &= \frac{n-n^3}{n^2-1} = -n,\\
-\frac{n}{n+1}\left(n-1+\frac{n}{n-1}\right)-\frac{n(n-2)}{n^2-1} &= \frac{-n[(n-1)^2+n]-n(n-2)}{n^2-1} = -n,
\end{aligned}$$

the above expression can be simplified as

$$\begin{aligned}
\mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi) &= -\mathcal{L}^{\mathrm{M}}_{\omega}(\varphi)+\frac{1}{V_\omega}\int_X\varphi\omega^n\\
&- \frac{n}{2V_\omega}\sum_{i=1}^{n-2}\int_X\varphi\omega^i_\varphi\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi-\frac{n}{2V_\omega}\int_X\varphi\omega^{n-2}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
&+ \frac{n}{2V_\omega}\sum_{i=1}^{n-2}\int_X\varphi\omega^i_\varphi\wedge\omega^{n-2-i}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi+\frac{n}{2V_\omega}\int_X\varphi\omega^{n-2}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi.
\end{aligned}$$

In summary,

Proposition C.2. *If $n\geq 3$, one has*

$$\begin{aligned}
\mathcal{J}^{\mathrm{AY}}_{\omega}(\varphi) &= -\mathcal{L}^{\mathrm{M}}_{\omega}(\varphi)+\frac{1}{V_\omega}\int_X\varphi\omega^n\\
&- \frac{n}{2V_\omega}\sum_{i=1}^{n-2}\int_X\varphi\omega^i_\varphi\wedge\omega^{n-2-i}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi-\frac{n}{2V_\omega}\int_X\varphi\omega^{n-2}\wedge\sqrt{-1}\partial\omega\wedge\overline{\partial}\varphi\\
&+ \frac{n}{2V_\omega}\sum_{i=1}^{n-2}\int_X\varphi\omega^i_\varphi\wedge\omega^{n-2-i}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi+\frac{n}{2V_\omega}\int_X\varphi\omega^{n-2}\wedge\sqrt{-1}\overline{\partial}\omega\wedge\partial\varphi.
\end{aligned}$$

References

1. Chen, X. X., Tian, G., *Ricci flow on Kähler-Einstein surfaces*, Invent. Math. **147**(2002), no.3, 487–544.
2. Li, Y., *Mabuchi and Aubin-Yau functionals over complex surfaces*, preprint, arXiv:1002.3411, 2010.
3. Li, Y., *Mabuchi and Aubin-Yau functionals over complex three-folds*, preprint, arXiv:1003.5307, 2010.
4. Mabuchi, T., *K-energy maps integrating Futaki invatiants*, Tohoku Math. Journ., **38**(1986), 575–593.
5. Phong, D.H., Sturm, J., *Lectures on stability and constant scalar curvature*, preprint, arXiv: 0801.4179v2, 2008.
6. Yau, S.-T., *Review of geometry and analysis*, Kodaira's issue. Asian J. Math. **4**(2000), no.1, 235–278

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